On adding a variable to a Frobenius manifold and generalizations

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August 28, 2012

Abstract: Let $\pi : V \rightarrow M$ be a (real or holomorphic) vector bundle whose base has an almost Frobenius structure $(\circ_M, e_M, g_M)$ and typical fiber has the structure of a Frobenius algebra $(\circ_V, e_V, g_V)$. Using a connection D on the bundle V and a morphism $\alpha : V \rightarrow TM$, we construct an almost Frobenius structure $(\circ, e_V, g)$ on the manifold V and we study when it is Frobenius. In particular, we describe all (real) positive definite Frobenius structures on V obtained in this way, when M is a semisimple Frobenius manifold with non-vanishing rotation coefficients. In the holomorphic setting, we add a real structure k_M on M and a real structure k_V on the fibers of π and we study when an induced real structure on V, together with the almost Frobenius structure $(\circ, e_V, g)$, satisfy the tt^*-equations. Along the way, we prove various properties of adding variables to a Frobenius manifold, in connection with Legendre transformations and tt^*-geometry.

Key words: F-manifolds, Frobenius manifolds, tt^*-equations, Saito bundles, Legendre transformations.

MSC Classification: 53D45, 53B50.

1 Introduction

Frobenius manifolds were defined by B. Dubrovin [3], as a geometrical interpretation of the so called WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations. They also appear in many different areas of mathematics - singularity theory [6], quantum cohomology [9] and integrable systems [7], providing an unexpected link between these apparently different fields. Rather than the original definition of Dubrovin, along the paper we shall use the follow-

ing alternative definition of Frobenius manifolds (the equivalence of the two definitions is a consequence of Theorem 2.15 and Lemma 2.16 of [6]).

Definition 1. *1) An almost Frobenius structure on a manifold M in one of the standard categories (smooth or holomorphic) is given by a (fiber preserving) commutative, associative multiplication $\circ$ on TM, with unit field e, and a metric g on M, invariant with respect to $\circ$, i.e. such that*

$$g(X \circ Y, Z) = g(X, Y \circ Z) \quad \forall X, Y, Z \in \mathcal{X}(M).$$

2) An almost Frobenius structure $(\circ, e, g)$ on M is called Frobenius (and $(M, \circ, e, g)$ is a Frobenius manifold) if the following conditions hold:

i) $(M, \circ, e)$ is an F-manifold, that is,

$$L_{X\circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ), \quad \forall X, Y \in \mathcal{X}(M).$$

ii) the metric g is admissible on the F-manifold $(M, \circ, e)$ (i.e. the unit field e is parallel with respect to the Levi-Civita connection of g) and is flat.

In flat coordinates (t^i) for the Frobenius metric g one may write

$$g\left(\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k}\right) = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k},$$

for a function F, called the potential of the Frobenius manifold and defined up to adding a quadratic expression in t^i. The asssociativity of $\circ$ translates into the WDVV-equations for F: for any fixed i, j, k, r,

$$\sum_{m,s} \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} g^{sm} \frac{\partial^3 F}{\partial t^m \partial t^k \partial t^r} = \sum_{m,s} \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^s} g^{sm} \frac{\partial^3 F}{\partial t^m \partial t^i \partial t^r}.$$

In examples arising from singularity theory, a Frobenius manifold comes naturally equipped with an Euler field.

Definition 2. *An Euler field on a Frobenius manifold $(M, \circ, e, g)$ is a vector field E such that*

$$L_E(\circ) = \circ, \quad L_E(g) = dg$$

where d is a constant (equal to 2, when $g(e, e) \neq 0$).

Outline of the paper. It is customary in modern mathematics to consider various geometrical structures on manifolds and to search for similar structures on related manifolds (like submanifolds, quotient manifolds, total spaces of vector bundles, etc). The starting point of this paper is a result

proved in Chapter VII of [10], which states that there is a natural Frobenius structure on the product $M \times \mathbb{K}$ when $(M, \circ, e, g)$ is Frobenius ($\mathbb{K} = \mathbb{R}$ when M is a real manifold and $\mathbb{K} = \mathbb{C}$ when M is complex). It is usually referred as *the Frobenius structure obtained by adding a variable to* M, and has unit field $\frac{\partial}{\partial \tau}$ (where τ is the coordinate on $\mathbb{K}$). It is important to note that the Frobenius metric on $M \times \mathbb{K}^r$ is not a product metric, and neither the multiplication is a product multiplication. If E is an Euler field on M with $L_E(g) = 2g$, then $E + R$ (where $R_{(p,\tau)} = \tau \frac{\partial}{\partial \tau}$, $(p, \tau) \in M \times \mathbb{K}$ is the radial field) is Euler on $M \times \mathbb{K}$. The overall aim of this paper is to develop generalizations of this construction.

Section 2, intended mostly for completeness (only Proposition 6 is original) contains a brief account of the Frobenius structure (with or without an Euler field) obtained from a vector bundle with additional data (e.g. a Saito structure) and a primitive section [12]. Such a data appears naturally in the theory of isomonodromic deformations. Then we recall, following [10], the construction of adding a variable to a Frobenius manifold. With this preliminary background, we determine the general form of the Frobenius structure on $M \times \mathbb{K}^r$ ($r \geq 1$), obtained by Legendre transformations and successively adding variables to a Frobenius manifold M (see Proposition 6), this being a good motivation for our treatment from the following sections.

The general set-up of the paper is a $\mathbb{K}$-vector bundle $\pi : V \to M$, with a connection D, an associative, commutative multiplication $\circ_M$ on TM, with unit field e_M, a similar multiplication $\circ_V$ on the fibers of V, with unit field $e_V \in \Gamma(V)$, and a morphism $\alpha : V \to TM$. From this data we construct a multiplication $\circ$ on TV, with unit field e_V (viewed as a tangent vertical vector field). Later on, we add to this picture an invariant metric g_M on $(M, \circ_M, e_M)$ and an invariant metric g_V on the fibers of V, and we construct a metric g on the manifold V. (The case when V is the trivial bundle of rank one, D is the trivial connection, $\alpha(e_V) = e_M$ and $(M, \circ_M, e_M, g_M)$ is Frobenius corresponds to adding a variable to M). We determine various properties of the structures on V so defined, as follows.

In Section 3 we prove that $(V, \circ, e_V)$ is an F-manifold if and only if $(M, \circ_M, e_M)$ is an F-manifold, the connection D is flat, the multiplication $\circ_V$ is D-parallel and two other natural conditions are satisfied, namely conditions (26) and (27) (see Proposition 7). We remark that the commutativity condition (27) is implied by (26), when the F-manifold $(M, \circ_M, e_M)$ is semisimple (see Remark 8).

In Section 4 we assume that $(V, \circ, e_V)$ is an F-manifold and we prove that g is an admissible metric on $(V, \circ, e_V)$ if and only if $D(g_V) = 0$ and the coidentity $\epsilon_M := g_M(e_M, \cdot)$ is closed (see Proposition 9).

In Section 5 we assume that $(V, \circ, e_V, g)$ is an F-manifold with admissi-

ble metric and that the base $(M, \circ_M, e_M, g_M)$ is a Frobenius manifold and we show that the flatness of g (i.e. the only remaining condition for $(V, \circ, e_V, g)$ to be Frobenius) can be expressed in terms of a system of conditions for the morphism α, see Proposition 13. Our main result in this section is Theorem 14, which provides a complete description, in the real case, of all Frobenius structures $(\circ, e_V, g)$ on V, with positive definite metric g, when $(M, \circ_M, e_M, g_M)$ is a semisimple Frobenius manifold with non-vanishing rotation coefficients. It turns out that

$$\alpha = \lambda \otimes e_M \tag{1}$$

with $\lambda \in \Gamma(V^*)$ a D-parallel section satisfying

$$\lambda(e_V) = 1, \quad \lambda(v_1 \circ_V v_2) = \lambda(v_1)\lambda(v_2), \quad \forall v_1, v_2 \in V.$$

It would be interesting to understand if the morphism α must be of this form also in other signatures, or in the complex case. At the end of this section we return to the general picture (no semisimplicity assumptions) and we consider a class of Frobenius structures $(\circ, e_V, g)$ on the total space of the trivial bundle $\pi : V = M \times \mathbb{K}^r \to M$ obtained from the data $(D, \circ_M, e_M, \circ_V, e_V, \alpha, g_M, g_V)$ as before, where the base $(M, \circ_M, e_M, g_M)$ is a Frobenius manifold, D is the trivial standard connection and α is of the form (1) (see Theorem 15). We study in detail this class of Frobenius manifolds, in connection with Euler fields and Saito bundles (see Proposition 18) and we compute their potential (see Remark 17).

In Section 6 we return to the general setting of an holomorphic vector bundle $\pi : V \to M$ with data $(D, \circ_M, e_M, \circ_V, e_V, \alpha, g_M, g_V)$. Let $\circ$ and g be the associated multiplication and metric on the manifold V, defined by this data. (We do not assume that $(\circ, e_V, g)$ is a Frobenius structure). Instead, we assume that M has a real structure k_M and the fibers of π have a real structure k_V. Using k_M and k_V we define a real structure k on V (which coincides with k_M on the D-horizontal subbundle and with k_V on the vertical subbundle). Our main result is Theorem 23, which states the necessary and sufficient conditions for the tt^*-equations to hold on (V, ϕ, h) where $\phi_X Y = -X \circ Y$ and $h(X, Y) = g(X, kY)$. Then we discuss an example where all these conditions hold (see Example 27). Finally, we return to the Frobenius structure from Theorem 15 (with $\mathbb{K} = \mathbb{C}$) and we show that imposing the tt^*-equations in the framework of this section gives high restrictions on the geometry of the base $(M, \circ_M, e_M, g_M)$ (see Proposition 28 and the comments before this proposition).

Acknowledgements: This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project no. PN-II-ID-PCE-2011-3-0362.

2 Legendre transformations and adding variables to Frobenius manifolds

Along the paper we use the following conventions. A $\mathbb{K}$-manifold is a real manifold when $\mathbb{K} = \mathbb{R}$ or a complex manifold when $\mathbb{K} = \mathbb{C}$. We denote by $\mathcal{X}(M)$ the sheaf of $\mathbb{K}$-vector fields on a $\mathbb{K}$-manifold M (i.e. smooth vector fields, if M is a real manifold, or holomorphic vector fields, when M is a complex manifold). By a $\mathbb{K}$-vector bundle we mean a real vector bundle ($\mathbb{K} = \mathbb{R}$) or an holomorphic vector bundle ($\mathbb{K} = \mathbb{C}$). Unless otherwise stated, all objects we consider on $\mathbb{C}$-vector bundles (e.g. sections, metrics, connections, Higgs fields, endomorphisms, etc) are holomorphic and for a complex manifold M, TM will denote the holomorphic tangent bundle and $\Omega^1(M, V)$ the bundle of holomorphic 1-forms with values in a $\mathbb{C}$-vector bundle $V \to M$. In our conventions, the curvature R^D of a connection D is given by $R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$.

2.1 Frobenius structures from infinitesimal period mappings

Let $\pi : V \to M$ be a $\mathbb{K}$-vector bundle with $\mathrm{rank}(V) = \dim(M)$, endowed with a connection ∇, a metric g and a vector valued 1-form $\phi \in \Omega^1(M, \mathrm{End}V)$, satisfying the conditions:

$$R^{\nabla} = 0, \quad d^{\nabla}\phi = 0, \quad \nabla g = 0, \quad \phi \wedge \phi = 0, \quad \phi^* = \phi, \tag{2}$$

where

$$(\phi \wedge \phi)_{X,Y} := \phi_X \phi_Y - \phi_Y \phi_X, \quad X, Y \in TM$$

and, for any $X \in TM$, $\phi_X^* \in \mathrm{End}(V)$ is the adjoint of $\phi_X \in \mathrm{End}(V)$ with respect to g. Assume, moreover, that there is a vector field e on M such that $\phi_e = -\mathrm{Id}_V$ (where Id_V is the identity endomorphism of V). Let $\omega \in \Gamma(V)$ (usually called a primitive section) be ∇-parallel such that the map

$$\psi^{\omega} : TM \to V, \quad \psi^{\omega}(X) = -\phi_X(\omega) \tag{3}$$

is an isomorphism. Define a multiplication $\circ$ on TM, with unit field e, by

$$X \circ Y = (\psi^{\omega})^{-1} (\phi_X \phi_Y \omega).$$

Using ψ^ω, we may transport the metric g and the connection ∇ to a metric g^ω and a connection ∇^ω on TM. It is easy to see that $\circ$ is independent of the choice of primitive section, while for g^ω and ∇^ω the choice of ω is essential. As proved by K. Saito [12], $(M, \circ, e, g^\omega)$ is a Frobenius $\mathbb{K}$-manifold, with Levi-Civita connection ∇^ω.

Remark 3. In examples coming from singularity theory, the bundle V comes equipped with two additional endomorphisms, R_0 and R_∞, satisfying the conditions:

$$\begin{aligned}&\nabla R_0 + \phi = [\phi, R_\infty], \quad [R_0, \phi] = 0, \quad R_0^* = R_0;\\ &\nabla R_\infty = 0, \quad R_\infty^* + R_\infty = -w\mathrm{Id}_V,\end{aligned}$$

where $w \in \mathbb{K}$. If ω is homogeneous, i.e. $R_\infty(\omega) = -q\omega$ for $q \in \mathbb{K}$, then

$$E^\omega := (\psi^\omega)^{-1}(R_0(\omega))$$

is Euler for $(M, \circ, e, g^\omega)$, with $L_{E^\omega}(g^\omega) = (2(1+q) - w)\, g^\omega$, and

$$R_\infty^\omega := (\psi^\omega)^{-1} \circ R_\infty \circ \psi^\omega = \nabla^\omega(E^\omega) - (1+q)\mathrm{Id}.$$

The data $(\nabla, \phi, g, R_0, R_\infty)$ is usually called a Saito structure (of weight w) on V.

2.2 Adjunction of a variable to a Frobenius manifold

Following [10], we now recall that if $(M, \circ, e, g)$ is a Frobenius $\mathbb{K}$-manifold, then $M \times \mathbb{K}$ is also Frobenius, with unit field $\frac{\partial}{\partial \tau}$ (where τ is the coordinate on $\mathbb{K}$). In order to explain this statement, consider the direct sum bundle $TM \oplus L$, where

$$\pi : L = M \times \mathbb{K} \to M \tag{4}$$

is the trivial rank one bundle. Fix $v \in \mathbb{K}$ non-zero (seen as a constant section of L) and define a metric $\tilde{g}$ on the bundle L by $\tilde{g}(v, v) = 1$. Let D be the standard trivial connection on L and ∇ the Levi-Civita connection of g. Consider the Higgs field

$$\phi_X(Y) := -X \circ Y, \quad X, Y \in TM,$$

trivially extended, as a 1-form on M with values in $\mathrm{End}(TM \oplus L)$. It is easy to check that the data

$$(\nabla^L := \nabla \oplus D, \quad g^L := g \oplus \tilde{g}, \quad \phi^L := \phi) \tag{5}$$

on $TM \oplus L$ satisfies relations (2), and so does the data

$$\left(\pi^*(\nabla^L), \quad \pi^*(g^L), \quad \phi' := \pi^*(\phi^L) - d\tau \otimes \mathrm{Id}_{\pi^*(TM\oplus L)}\right) \tag{6}$$

on $\pi^*(TM \oplus L)$. Any Legendre field X_0 on $(M, \circ, e, g)$ (i.e. a parallel invertible vector field) determines a primitive section $\omega := \pi^*(X_0 + v)$ for $(\pi^*(TM \oplus L), \pi^*(\nabla^L), \pi^*(g^L), \phi')$. The isomorphism

$$\psi^\omega : T(M \times \mathbb{K}) \to \pi^*(TM \oplus L), \quad \psi^\omega(Z) = -\phi'_Z(\omega)$$

is given by

$$\psi^\omega(X) = \pi^*(X \circ X_0), \quad \psi^\omega\left(\frac{\partial}{\partial \tau}\right) = \pi^*(X_0 + v), \quad X \in TM \tag{7}$$

and the induced Frobenius structure $(\circ^{(1)}, \frac{\partial}{\partial \tau}, g^{(1)})$ on $M \times \mathbb{K}$ can be described as follows:

$$X \circ^{(1)} Y = X \circ Y, \quad X \circ^{(1)} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} \circ^{(1)} X = X, \quad \frac{\partial}{\partial \tau} \circ^{(1)} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} \tag{8}$$

for any $X, Y \in TM$ and

$$g^{(1)} = g^{X_0} + d\tau \otimes i_e(g^{X_0}) + i_e(g^{X_0}) \otimes d\tau + (g^{X_0}(e, e) + 1) d\tau \otimes d\tau, \tag{9}$$

where

$$g^{X_0}(X, Y) := g(X_0 \circ X, X_0 \circ Y)$$

is the Legendre transformation of g by the Legendre field X_0. The Frobenius structure on $M \times \mathbb{K}$ constructed in this way, with $X_0 = e$, is usually called the Frobenius structure obtained from $(M, \circ, e, g)$ by adding a variable ([10], Chapter VII).

Remark 4. If X_0 is a Legendre field on a Frobenius manifold $(M, \circ, e, g)$, then $(M, \circ, e, g^{X_0})$ is also Frobenius (see e.g. [4]). From (8) and (9) we deduce that the Frobenius structure on $M \times \mathbb{K}$, obtained as above by using $\pi^*(X_0 + v)$ as primitive section, coincides with the Frobenius structure obtained by adding a variable to the Legendre transformation $(M, \circ, e, g^{X_0})$ of $(M, \circ, e, g)$.

Remark 5. Assume now that the Frobenius manifold $(M, \circ, e, g)$ has an Euler field E such that $L_E(g) = dg$, for $d \in \mathbb{K}$, and let

$$\left(\nabla, \quad g, \quad \phi, \quad R_0 := -\phi_E, \quad R_\infty := \nabla E - \frac{1}{2}(w + d)\,\mathrm{Id}_{TM}\right)$$

be the associated Saito structure on TM of weight w, where ∇ is the Levi-Civita connection of g and $\phi_X(Y) := -X \circ Y$ is the Higgs field of the Frobenius manifold. Consider R_0 and R_∞ as endomorphisms of $TM \oplus L$, acting trivially on L. Then

$$\left(\pi^*(\nabla^L), \quad \pi^*(g^L), \quad \phi', \quad R'_\infty, \quad R'_0\right), \tag{10}$$

where π is the projection (4), ∇^L, g^L and ϕ' are defined as before by (5), (6), and

$$R'_\infty := \pi^*\left(R_\infty - \frac{w}{2}\mathrm{Id}_V\right), \quad R'_0 := \pi^*(R_0) + \tau\mathrm{Id}_{\pi^*(TM\oplus L)},$$

is a Saito structure of weight w on $\pi^*(TM \oplus L)$. Remark that $\pi^*(e+v)$ is $\pi^*(\nabla^L)$-flat, and, moreover, it is an eigenvector of R'_∞ if and only if $d = 2$. Therefore, if $d = 2$, $\omega := \pi^*(e+v)$ is a primitive homogeneous section for the Saito bundle $\pi^*(TM \oplus L)$ with data (10). The induced Frobenius structure on $M \times \mathbb{K}$ has an Euler field, namely $E + R$, where $R_{(p,\tau)} := \tau\frac{\partial}{\partial\tau}$ is the radial field. This is because

$$E + R = (\psi^\omega)^{-1} R'_0\left(\pi^*(e+v)\right),$$

see Remark 3 and (7) (with $X_0 = e$).

2.3 Iterations

In this section we determine the general form of the Frobenius structure obtained by Legendre transformations and successively adding variables to a Frobenius manifold, as follows.

Proposition 6. *The Frobenius structure on $M \times \mathbb{K}^r$ ($r \geq 1$) obtained from a Frobenius $\mathbb{K}$-manifold $(M, \circ, e, g)$ by Legendre transformations and adding variables has multiplication $\circ^{(r)}$ given by*

$$X \circ^{(r)} Y = X \circ Y, \quad X \circ^{(r)} \frac{\partial}{\partial\tau_i} = X, \quad \frac{\partial}{\partial\tau_i} \circ^{(r)} \frac{\partial}{\partial\tau_j} = \frac{\partial}{\partial\tau_{\min\{i,j\}}} \tag{11}$$

for any $X, Y \in TM$, $1 \leq i, j \leq r$, unit field $\frac{\partial}{\partial\tau^r}$ and metric

$$g^{(r)} = g^{Z_0} + \sum_{k=1}^{r}\left(d\tau_k \otimes i_e(g^{Z_0}) + i_e(g^{Z_0}) \otimes d\tau_k\right) + \sum_{i,j=1}^{r} g^{(r)}_{ij} d\tau_i \otimes d\tau_j, \tag{12}$$

where Z_0 is a Legendre field on $(M, \circ, e, g)$ and $g^{(r)}_{ij}$ are constants, satisfying

$$g^{(r)}_{ij} = g^{(r)}_{ji} = g^{(r)}_{\min\{i,j\},r}, \quad \forall 1 \leq i, j \leq r. \tag{13}$$

Above $(\tau_1, \cdots, \tau_r)$ is the standard coordinate system of $\mathbb{K}^r$.

Proof. We only prove the statement for $r = 2$, the general case being similar. Let X_0 be a Legendre field on $(M, \circ, e, g)$ and $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau_1}, g^{(1)})$ the Frobenius manifold obtained by adding a variable to the Legendre transformation $(M, \circ, e, g^{X_0})$ of $(M, \circ, e, g)$. Thus, from relations (8) and (9) of Section 2.2, $\circ^{(1)}$ and $g^{(1)}$ are given by

$$X \circ^{(1)} Y = X \circ Y, \quad X \circ^{(1)} \frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \tau_1} \circ^{(1)} X = X, \quad \frac{\partial}{\partial \tau_1} \circ^{(1)} \frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \tau_1} \tag{14}$$

for any $X, Y \in TM$ and

$$g^{(1)} = g^{X_0} + d\tau_1 \otimes i_e(g^{X_0}) + i_e(g^{X_0}) \otimes d\tau_1 + (g^{X_0}(e, e) + 1) d\tau_1 \otimes d\tau_1. \tag{15}$$

Let $(M \times \mathbb{K}^2, \circ^{(2)}, \frac{\partial}{\partial \tau_2}, g^{(2)})$ be the Frobenius manifold obtained by adding a variable to $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau_1}, (g^{(1)})^Z)$, where Z is a Legendre field on $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau_1}, g^{(1)})$. We need to check that $\circ^{(2)}$ and $g^{(2)}$ are given by (11) and (12) (with $r = 2$), for a Legendre field Z_0 on $(M, \circ, e, g)$ which needs to be determined. It is easy to check the statement for $\circ^{(2)}$. We only prove the statement for $g^{(2)}$. In order to prove that $g^{(2)}$ is of the required form, we first notice that Z, being a Legendre field on $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau^1}, g^{(1)})$, it is parallel with respect to the Levi-Civita connection of $g^{(1)}$ and therefore it decomposes into a sum

$$Z = Z^{TM} + c \frac{\partial}{\partial \tau^1}$$

where Z^{TM} is a vector field on M which is parallel with respect to the Levi-Civita connection of g^{X_0} and $c \in \mathbb{K}$ (since $g^{(1)}$ is given by (15), any parallel vector field on $(M \times \mathbb{K}, g^{(1)})$ is a sum of a parallel vector field on (M, g^{X_0}) and a constant multiple of $\frac{\partial}{\partial \tau^1}$ - this can be checked directly; see also Remark 11).

With this preliminary remark, we now claim that $g^{(2)}$ is given by (12), with

$$Z_0 := X_0 \circ (Z^{TM} + ce), \tag{16}$$

and that Z_0 is a Legendre field on $(M, \circ, e, g)$. We divide the proof of this claim in two steps: first, we prove that Z_0 is Legendre on $(M, \circ, e, g)$; next, we prove that $g^{(2)}$ is of the form (12), with Z_0 given by (16).

To prove that Z_0 given by (16) is a Legendre field on $(M, \circ, e, g)$, we recall that the Levi-Civita connections of g and g^{X_0} are related by a Legendre transformation

$$\nabla^{g^{X_0}} = X_0^{-1} \circ \nabla^g \circ X_0.$$

Therefore, since Z^{TM} is $\nabla^{g^{X_0}}$-parallel, so is $Z^{TM} + ce$ (because $\nabla^g X_0 = 0$) and Z_0, defined by (16), is parallel with respect to ∇^g. Moreover, it may be

checked that if $W \in \mathcal{X}(M \times \mathbb{K})$ is the inverse of Z with respect to $\circ^{(1)}$ (which exists, because Z is Legendre on $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau_1}, g^{(1)})$), then the projection W^{TM} of W to TM is a vector field on M and

$$\left(W^{TM} + \frac{1}{c} e\right) \circ \left(Z^{TM} + ce\right) = e,$$

(note also that $c \neq 0$), i.e. $Z^{TM} + ce$, hence also Z_0, are invertible with respect to $\circ$. We conclude that Z_0 is a Legendre field on $(M, \circ, e, g)$, as required.

It remains to prove that $g^{(2)}$ is given by (16). For this, we use relations (8), (9) and that $\frac{\partial}{\partial \tau^1}$ is the unit field for $\circ^{(1)}$ and we get:

$$g^{(2)} = (g^{(1)})^Z + d\tau_2 \otimes i_{\frac{\partial}{\partial \tau^1}} (g^{(1)})^Z + i_{\frac{\partial}{\partial \tau^1}} (g^{(1)})^Z \otimes d\tau_2 + \left(g^{(1)}(Z, Z) + 1\right) d\tau_2 \otimes d\tau_2. \tag{17}$$

From (14), (15) and (17), for any $X, Y \in TM$,

$$g^{(2)}(X, Y) = g^{(1)}(Z \circ^{(1)} X, Z \circ^{(1)} Y) = g^{X_0}\left((Z^{TM} + ce)^2 \circ X, Y\right), \tag{18}$$

because

$$Z \circ^{(1)} X = \left(Z^{TM} + c\frac{\partial}{\partial \tau^1}\right) \circ^{(1)} X = Z^{TM} \circ X + cX = (Z^{TM} + ce) \circ X. \tag{19}$$

Thus,

$$g^{(2)}(X, Y) = g(Z_0 \circ X, Z_0 \circ Y) = g^{Z_0}(X, Y), \quad \forall X, Y \in TM.$$

From (15), (17) and (19),

$$\begin{aligned}
g^{(2)}(\frac{\partial}{\partial \tau^1}, Y) &= (g^{(1)})^Z \left(\frac{\partial}{\partial \tau_1}, Y\right) = g^{(1)}(Z \circ^{(1)} \frac{\partial}{\partial \tau^1}, Z \circ^{(1)} Y) \\
&= g^{(1)}(Z, (Z^{TM} + ce) \circ Y) = g^{(1)}(Z^{TM} + c\frac{\partial}{\partial \tau_1}, (Z^{TM} + ce) \circ Y) \\
&= g^{X_0}\left((Z^{TM} + ce)^2, Y\right) = g^{Z_0}(e, Y)
\end{aligned}$$

where in the second line we used (19) and that $\frac{\partial}{\partial \tau^1}$ is the unit field for $\circ^{(1)}$, and in the last line we used the definition (15) of $g^{(1)}$. The above computation also implies

$$g^{(2)}(\frac{\partial}{\partial \tau^2}, Y) = (g^{(1)})^Z \left(\frac{\partial}{\partial \tau_1}, Y\right) = g^{Z_0}(e, Y), \quad \forall Y \in TM.$$

We proved that $g^{(2)}$ is of the form (12), with $r = 2$, Z_0 given by (16) and coefficients

$$g^{(2)}_{ij} := g^{(2)}\left(\frac{\partial}{\partial \tau^i}, \frac{\partial}{\partial \tau^j}\right) = g^{(2)}\left(\frac{\partial}{\partial \tau^i} \circ^{(2)} \frac{\partial}{\partial \tau^j}, \frac{\partial}{\partial \tau^r}\right) = g^{(2)}_{\min\{i,j\}r}, \quad \forall 1 \leq i, j \leq 2,$$

because $\frac{\partial}{\partial \tau_i} \circ^{(2)} \frac{\partial}{\partial \tau_j} = \frac{\partial}{\partial \tau_{\min\{i,j\}}}$. Remark that $g^{(2)}_{ij}$ are constant, because

$$g^{(2)}\left(\frac{\partial}{\partial \tau^1}, \frac{\partial}{\partial \tau^1}\right) = g^{(2)}\left(\frac{\partial}{\partial \tau^1}, \frac{\partial}{\partial \tau^2}\right) = g^{(1)}(Z, Z)$$
$$g^{(2)}\left(\frac{\partial}{\partial \tau^2}, \frac{\partial}{\partial \tau^2}\right) = g^{(1)}(Z, Z) + 1$$

and Z is Legendre (hence, parallel) on $(M \times \mathbb{K}, \circ^{(1)}, \frac{\partial}{\partial \tau^1}, g^{(1)})$.

□

In the following sections we will search for more general Frobenius structures on total spaces of vector bundles.

3 Total spaces of vector bundles as F-manifolds

Here and in the following sections we fix a $\mathbb{K}$-vector bundle $\pi : V \to M$ with the following additional data:

- a **connection D on V**, which induces a decomposition

$$T_v V = T_p M \oplus V_p, \quad \forall v \in V \tag{20}$$

into horizontal and vertical subspaces. The horizontal lift of a vector field $X \in \mathcal{X}(M)$ will be denoted $\tilde{X}$. When a longer term *Expression* is lifted to V, the lift will be denoted by $[Expression]\tilde{}$. Often sections of V will be considered (without mentioning explicitly) as vertical vector fields on the manifold V. Recall that

$$[\tilde{X}, \tilde{Y}]_v = [X, Y]\tilde{}_v - R^D_{X,Y} v, \quad \forall X, Y \in \mathcal{X}(M), \quad \forall v \in V \tag{21}$$

and

$$[\tilde{X}, s] = D_X s, \quad \forall X \in \mathcal{X}(M), \quad \forall s \in \Gamma(V). \tag{22}$$

- a (fiber preserving) **multiplication $\circ_M$ on TM**, which is commutative, associative, with unit $e_M \in \mathcal{X}(M)$;

- a (fiber preserving) **multiplication $\circ_V$ on the bundle V**, which is commutative, associative, with unit $e_V \in \Gamma(V)$;

- A **bundle morphism**

$$\Theta : TM \oplus V \to TM, \quad (X, v) \to \Theta(X, v),$$

such that

$$\Theta(X, e_V) = X, \quad \forall X \in TM. \tag{23}$$

We construct from the data $(D, \circ_M, \circ_V, \Theta)$ a (fiber preserving) multiplication $\circ$ on TV, as follows:

$$\tilde{X} \circ \tilde{Y} := [X \circ_M Y]\tilde{}, \quad v_1 \circ v_2 := v_1 \circ_V v_2, \quad v \circ \tilde{X} = \tilde{X} \circ v := [\Theta(X, v)]\tilde{}$$

where $X, Y \in T_pM$ and $v_1, v_2 \in V_p$. From (23), e_V is the unit field for $\circ$. We also require that $\circ$ is associative and commutative, which is equivalent to

$$\Theta(X, v) = X \circ_M \alpha(v), \quad \forall X \in TM, \quad \forall v \in V,$$

with

$$\alpha : V \to TM, \quad \alpha(v) := \Theta(e_M, v)$$

satisfying

$$\alpha(v_1 \circ_V v_2) = \alpha(v_1) \circ_M \alpha(v_2), \quad \alpha(e_V) = e_M. \tag{24}$$

Thus, $\circ$ is given by

$$\tilde{X} \circ \tilde{Y} := [X \circ_M Y]\tilde{}, \quad v_1 \circ v_2 := v_1 \circ_V v_2, \quad v \circ \tilde{X} := \tilde{X} \circ v = [\alpha(v) \circ_M X]\tilde{}. \tag{25}$$

Our main result from this section is the following.

Proposition 7. *The multiplication $\circ$ defines an F-manifold structure on V (with unit field e_V) if and only if the following conditions are satisfied:*

1) $(M, \circ_M, e_M)$ is an F-manifold;

2) the connection D is flat;

3) For any D-parallel (local) section $s \in \Gamma(V)$,

$$L_{\alpha(s)}(\circ_M) = 0. \tag{26}$$

4) If $s_1, s_2 \in \Gamma(V)$ are D-parallel, then so is $s_1 \circ_V s_2$ and, moreover,

$$[\alpha(s_1), \alpha(s_2)] = 0. \tag{27}$$

Proof. We need to show that

$$L_{\mathcal{W}_1 \circ \mathcal{W}_2}(\circ)(\mathcal{W}_3, \mathcal{W}_4) = \mathcal{W}_1 \circ L_{\mathcal{W}_2}(\circ)(\mathcal{W}_3, \mathcal{W}_4) + \mathcal{W}_2 \circ L_{\mathcal{W}_1}(\circ)(\mathcal{W}_3, \mathcal{W}_4) \tag{28}$$

for any vector fields $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4 \in \mathcal{X}(V)$, is equivalent to the conditions from the statement of the proposition. Suppose first that (28) holds and

let $\mathcal{W}_1 = \tilde{X}$, $\mathcal{W}_2 = \tilde{Y}$, $\mathcal{W}_3 = \tilde{Z}$, $\mathcal{W}_4 = \tilde{T}$ be D-horizontal lifts, where $X, Y, Z, T \in \mathcal{X}(M)$. Using (21) and the definition of $\circ$, one can show that

$$L_{\tilde{X}\circ\tilde{Y}}(\circ)(\tilde{Z}, \tilde{T}) = \tilde{X} \circ L_{\tilde{Y}}(\circ)(\tilde{Z}, \tilde{T}) + \tilde{Y} \circ L_{\tilde{X}}(\circ)(\tilde{Z}, \tilde{T})$$

if and only if

$$R^D_{X\circ Y, Z\circ T} = 0$$

and

$$L_{X\circ Y}(\circ_M)(Z, T) = X \circ L_Y(\circ_M)(Z, T) + Y \circ L_X(\circ_M)(Z, T),$$

i.e. D is flat and $(M, \circ_M, e_M)$ is an F-manifold. Thus, the first two conditions from the proposition hold. Next, let $\mathcal{W}_1 := \tilde{X}$, $\mathcal{W}_2 = s \in \Gamma(V)$, $\mathcal{W}_3 = \tilde{Z}$ and $\mathcal{W}_4 = \tilde{T}$. Using (21) (with $R^D = 0$) and (22), one can check that

$$L_{\tilde{X}\circ s}(\circ)(\tilde{Z}, \tilde{T}) = \tilde{X} \circ L_s(\circ)(\tilde{Z}, \tilde{T}) + s \circ L_{\tilde{X}}(\circ)(\tilde{Z}, \tilde{T})$$

if and only if

$$L_{\alpha(s)}(\circ_M)(Z, T) = -\alpha(D_{Z\circ_M T}s) + T \circ_M \alpha(D_Z s) + Z \circ_M \alpha(D_T s), \qquad (29)$$

which is equivalent to (26) (using that D is flat). Thus, the third condition holds as well. Consider now $\mathcal{W}_1 = s_1$, $\mathcal{W}_2 = s_2$, $\mathcal{W}_3 = \tilde{Z}$ and $\mathcal{W}_4 = \tilde{T}$. Using (22) again, one can show that

$$L_{s_1\circ s_2}(\circ)(\tilde{Z}, \tilde{T}) = s_1 \circ L_{s_2}(\circ)(\tilde{Z}, \tilde{T}) + s_2 \circ L_{s_1}(\circ)(\tilde{Z}, \tilde{T})$$

if and only if

$$D_Z(s_1 \circ_V s_2) = (D_Z s_1) \circ_V s_2 + s_1 \circ_V (D_Z s_2), \qquad (30)$$

which is equivalent to $D(s_1 \circ_V s_2) = 0$, when $D(s_i) = 0$, $i = 1, 2$ (again, because D is flat). Finally, let $\mathcal{W}_1 := \tilde{X}$, $\mathcal{W}_2 = s_2$, $\mathcal{W}_3 = \tilde{Z}$ and $\mathcal{W}_4 = s_4$. Using that $(M, \circ_M, e_M)$ is an F-manifold and (29), it can be shown that

$$L_{\tilde{X}\circ s_2}(\circ)(\tilde{Z}, s_4) = \tilde{X} \circ L_{s_2}(\circ)(\tilde{Z}, s_4) + s_2 \circ L_{\tilde{X}}(\circ)(\tilde{X}, s_4)$$

if and only if

$$[\alpha(s_2), \alpha(s_4)] = \alpha\left(D_{\alpha(s_2)}s_4 - D_{\alpha(s_4)}s_2\right),$$

i.e. the fourth condition holds as well. We have proved that if $(V, \circ, e_V)$ is an F-manifold, then all conditions from the proposition are satisfied. Conversely, assume that all conditions are satisfied. Our previous argument shows that (28) holds when all $\mathcal{W}_i$ are horizontal, or when one is vertical and the other three are horizontal, or when two are vertical and two are horizontal. It is easy to check that (28) holds also with the remaining type of arguments (i.e. three vertical and one horizontal, or all four vertical). Our claim follows. □

Remark 8. When $(M, \circ_M, e_M)$ is semisimple, i.e. there is a coordinate system $(u^1, \cdots, u^n)$ (called canonical) such that $\frac{\partial}{\partial u^i} \circ_M \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}$, for any i, j, condition (26) implies condition (27). The reason is that a vector field X on a semisimple F-manifold $(M, \circ_M, e_M)$ satisfies $L_X(\circ_M) = 0$ if and only if it is of the form $X = \sum_{k=1}^n a_k \frac{\partial}{\partial u^k}$, for some constants a_k (see e.g. [4]). Any two such vector fields commute.

4 Admissible metrics on $(V, \circ, e_V)$

We consider the setting of Section 3 and we assume that all conditions from Proposition 7 are satisfied, i.e. $(V, \circ, e_V)$ is an F-manifold. We now add to the picture a metric g_M on M, invariant with respect to $\circ_M$, and a metric g_V on the bundle V, invariant with respect to $\circ_V$. In the remaining part of the paper we assume that the $(2,0)$-tensor field on the manifold V, defined by

$$g(\tilde{X}, \tilde{Y}) := g_M(X, Y), \quad g(v_1, v_2) := g_V(v_1, v_2), \quad g(v, \tilde{X}) := g_M(\alpha(v), X), \tag{31}$$

for any $X, Y \in TM$ and $v, v_1, v_2 \in V$, is non-degenerate. This is equivalent to the non-degeneracy of the metric $g_V - \alpha^* g_M$ of the bundle V (easy check). Also, from (24) and the invariance of g_M and g_V, the metric g is invariant on $(V, \circ, e_V)$.

Proposition 9. *Assume that $(V, \circ, e_V)$ is an F-manifold. Then the metric g defined by (31) is admissible on $(V, \circ, e_V)$ if and only if $D(g_V) = 0$ and the coidentity $\epsilon_M := g_M(e_M, \cdot)$ is closed.*

Proof. We first remark that the unit section $e_V \in \Gamma(V)$ is D-parallel: this follows by taking the covariant derivative (with respect to D) of the equality $e_V \circ_V e_V = e_V$ and using relation (30). Thus,

$$[\tilde{X}, e_V] = D_X(e_V) = 0, \quad \forall X \in \mathcal{X}(M). \tag{32}$$

The Koszul formula for the Levi-Civita connection ∇ of g, together with (32), $\alpha(e_V) = e_M$ and $R^D = 0$, imply that for any $Y, Z \in \mathcal{X}(M)$,

$$g(\nabla_{\tilde{Y}}(e_V), \tilde{Z}) = \frac{1}{2}\left(g_M(\nabla^M_Y(\alpha(e_V)), Z) - g_M(\nabla^M_Z(\alpha(e_V)), Y)\right) = \frac{1}{2}(d\epsilon_M)(Y, Z),$$

where ∇^M is the Levi-Civita connection of g_M. Similarly,

$$g(\nabla_{\tilde{Y}}(e_V), s) = -g(\nabla_s(e_V), \tilde{Y}) = \frac{1}{2} D_Y(\epsilon_V)(s), \quad g(\nabla_s e_V, \tilde{s}) = 0,$$

for any $s, \tilde{s} \in \Gamma(V)$, where $\epsilon_V \in \Gamma(V^*)$ is the g_V-dual to e_V, i.e.

$$\epsilon_V(v) := g_V(e_V, v), \quad \forall v \in V.$$

Thus, $\nabla(e_V) = 0$ is equivalent to $D(\epsilon_V) = 0$ and $d\epsilon_M = 0$. Now, we claim that $D(\epsilon_V) = 0$ is equivalent to $D(g_V) = 0$. This is a consequence of the relation

$$g_V(s_1, s_2) = \epsilon_V(s_1 \circ_V s_2), \quad \forall s_1, s_2 \in \Gamma(V)$$

and the fact that if s_1 and s_2 are D-parallel, then so is $s_1 \circ_V s_2$ (from Proposition 7). Our claim follows.

□

5 Frobenius structures on V

We consider the almost Frobenius structure $(\circ, e_V, g)$ on V constructed from the data $(D, \circ_M, e_M, \circ_V, e_V, g_M, g_V)$ as in the previous section, and we assume that $(V, \circ, e_V, g)$ is an F-manifold with admissible metric (hence the conditions from Propositions 7 and 9 are satisfied) and the base $(M, \circ_M, e_M, g_M)$ is a Frobenius manifold. In this section we compute the curvature of g and we find conditions on the morphism α which insure that g is flat, or $(V, \circ, e_V, g)$ is Frobenius. Our main result is Theorem 14, which describes all real Frobenius structures $(\circ, e_V, g)$ on V, with positive definite metric g, when M is (real) semisimple with non-vanishing rotation coefficients.

We begin by computing the Levi-Civita connection of g.

Lemma 10. *The Levi-Civita connection ∇ of g is given by*

$$g(\nabla_{\tilde{Y}}\tilde{X}, \tilde{Z}) = g_M(\nabla^M_Y X, Z)$$
$$g(\nabla_{\tilde{Y}}\tilde{X}, s) = g_M(\nabla^M_Y X, \alpha(s)) + \frac{1}{2}\Big(g_M(\nabla^{M,D}_X(\alpha)(s), Y) + g_M(\nabla^{M,D}_Y(\alpha)(s), X)\Big)$$
$$g(\nabla_{s_1}\tilde{X}, s_2) = 0$$
$$g(\nabla_s \tilde{X}, \tilde{Y}) = \frac{1}{2}\Big(g_M(\nabla^{M,D}_X(\alpha)(s), Y) - g_M(\nabla^{M,D}_Y(\alpha)(s), X)\Big)$$
$$\nabla_{\tilde{X}} s = \nabla_s \tilde{X} + D_X s$$
$$\nabla_{s_1} s_2 = 0,$$

for any vector fields $X, Y, Z \in \mathcal{X}(M)$ and sections $s, s_1, s_2 \in \Gamma(V)$. Above ∇^M is the Levi-Civita connection of g_M, the morphism α is viewed as a section of $TM \otimes V^*$ and $\nabla^{M,D}$ is the product connection $\nabla^M \otimes D$ on $TM \otimes V^*$.

Proof. The proof is a straightforward computation, which uses the flatness of D (which implies $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]}$, for any $X, Y \in \mathcal{X}(M)$) and $D(g_V) = 0$ (from Proposition 9).

$\square$

Remark 11. In the above lemma, assume that $\nabla^{M,D}(\alpha) = 0$. Then a vector field is ∇-parallel if and only if it is of the form $\tilde{X} + s$, where X is ∇^M-parallel and s is a D-parallel section of V, viewed as a tangent vertical vector field on V; since g_M and D are flat, locally there is a maximal number of parallel vector fields on (V, g) and g is flat as well. This applies to the particular case when V is the trivial bundle of rank one and the almost Frobenius structure on V is given by adding a variable to $(M, \circ_M, e_M, g_M)$; in this case, $\alpha(e_V) = e_M$ and thus $\nabla^{M,D}(\alpha) = 0$ (because $\nabla^M(e_M) = 0$ and $D(e_V) = 0$).

Next, we compute the curvature of g. For this, we first introduce new notations, as follows. For a tangent vector $\mathcal{W} \in T_vV$, we denote by $\mathcal{W}^{\mathrm{hor}}$ and $\mathcal{W}^{\mathrm{vert}}$ its horizontal and vertical components with respect to the decomposition

$$T_vV = T_pM \oplus V_p$$

induced by the connection D. From the definition of g, a tangent vector $\mathcal{W} \in T_vV$ belongs to $(T_pM)^{\perp}$ (i.e. is g-orthogonal to $T_pM \subset T_vV$) if and only if

$$\mathcal{W} = -\alpha(\mathcal{W}^{\mathrm{vert}}) + \mathcal{W}^{\mathrm{vert}}. \tag{33}$$

Similarly, $W \in (V_p)^{\perp}$ (i.e. is g-orthogonal to $V_p \subset T_vV$) if and only if

$$g_M(\mathcal{W}^{\mathrm{hor}}, \alpha(w)) + g_V(\mathcal{W}^{\mathrm{vert}}, w) = 0, \quad \forall w \in V_p. \tag{34}$$

Lemma 12. *The curvature R^{∇} of the metric g has the following expression: for any ∇^M-parallel vector fields $X, Y, Z, T \in \mathcal{X}(M)$ and D-parallel sections $s, s_1, s_2 \in \Gamma(V)$,*

$$
\begin{aligned}
R^{\nabla}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{T}) &= (g_V-\alpha^*g_M)((\nabla_{\tilde{X}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{Y}}\tilde{T})^{\mathrm{vert}})\\
&\quad -(g_V-\alpha^*g_M)((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}});\\
R^{\nabla}(\tilde{X},\tilde{Y},\tilde{Z},s) &= \frac{1}{2}\left(g_M(\nabla^M_X[\nabla^M_Z(\alpha(s))],Y)-g_M(\nabla^M_Y[\nabla^M_Z(\alpha(s))],X)\right)\\
&\quad +(g_V-\alpha^*g_M)\left((\nabla_{\tilde{X}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{Y}}s)^{\mathrm{vert}}\right)\\
&\quad -(g_V-\alpha^*g_M)\left((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{X}}s)^{\mathrm{vert}}\right);\\
R^{\nabla}(\tilde{X},\tilde{Y},s_1,s_2) &= -g_M((\nabla_{\tilde{Y}}s_1)^{\mathrm{hor}},(\nabla_{\tilde{X}}s_2)^{\mathrm{hor}})+g_V((\nabla_{\tilde{X}}s_2)^{\mathrm{vert}},(\nabla_{\tilde{Y}}s_1)^{\mathrm{vert}})\\
&\quad +g_M((\nabla_{\tilde{X}}s_1)^{\mathrm{hor}},(\nabla_{\tilde{Y}}s_2)^{\mathrm{hor}})-g_V((\nabla_{\tilde{Y}}s_2)^{\mathrm{vert}},(\nabla_{\tilde{X}}s_1)^{\mathrm{vert}});\\
R^{\nabla}(\tilde{X},s_1,\tilde{Y},s_2) &= g_V((\nabla_{\tilde{X}}s_2)^{\mathrm{vert}},(\nabla_{s_1}\tilde{Y})^{\mathrm{vert}})-g_M((\nabla_{s_1}\tilde{Y})^{\mathrm{hor}},(\nabla_{\tilde{X}}s_2)^{\mathrm{hor}})\\
R^{\nabla}(s_1,s_2)(s) &= 0.
\end{aligned}
$$

Proof. We compute $R^{\nabla}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{T})$, where $X,Y,Z,T\in\mathcal{X}(M)$ are ∇^M-parallel, as follows:

$$
\begin{aligned}
R^{\nabla}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{T}) &= g\left(\nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z}-\nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z}-\nabla_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{T}\right)\\
&= \tilde{X}g\left(\nabla_{\tilde{Y}}\tilde{Z},\tilde{T}\right)-g\left(\nabla_{\tilde{Y}}\tilde{Z},\nabla_{\tilde{X}}\tilde{T}\right)\\
&\quad -\tilde{Y}g\left(\nabla_{\tilde{X}}\tilde{Z},\tilde{T}\right)+g\left(\nabla_{\tilde{X}}\tilde{Z},\nabla_{\tilde{Y}}\tilde{T}\right)\\
&= -g\left(\nabla_{\tilde{Y}}\tilde{Z},\nabla_{\tilde{X}}\tilde{T}\right)+g\left(\nabla_{\tilde{X}}\tilde{Z},\nabla_{\tilde{Y}}\tilde{T}\right),
\end{aligned}
$$

where we used that $\nabla_{\tilde{Y}}\tilde{Z}$, $\nabla_{\tilde{X}}\tilde{Z}$ are g-orthogonal to TM (from Lemma 10) and $[\tilde{X},\tilde{Y}]=0$, because $R^D=0$ and $[X,Y]=0$. Thus:

$$
R^{\nabla}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{T})=-g(\nabla_{\tilde{Y}}\tilde{Z},\nabla_{\tilde{X}}\tilde{T})+g(\nabla_{\tilde{X}}\tilde{Z},\nabla_{\tilde{Y}}\tilde{T}). \tag{35}
$$

On the other hand, since $\nabla_{\tilde{Y}}\tilde{Z}$ and $\nabla_{\tilde{X}}\tilde{T}$ are g-orthogonal to TM, from (33),

$$
(\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{hor}}=-\alpha((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}}) \tag{36}
$$

and similarly for $\nabla_{\tilde{X}}\tilde{T}$. We obtain:

$$
\begin{aligned}
g(\nabla_{\tilde{Y}}\tilde{Z},\nabla_{\tilde{X}}\tilde{T}) &= g(\nabla_{\tilde{Y}}\tilde{Z},(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{hor}}+(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}})\\
&= g\left((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{hor}}+(\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}}\right)\\
&= g_M((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{hor}},\alpha((\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}}))+g_V((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}})\\
&= (g_V-\alpha^*g_M)((\nabla_{\tilde{Y}}\tilde{Z})^{\mathrm{vert}},(\nabla_{\tilde{X}}\tilde{T})^{\mathrm{vert}}),
\end{aligned}
$$

where in the last equality we used (36). Combining this relation with (35) we obtain $R^{\nabla}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T})$, as required. The remaining expressions of the curvature can be obtained in a similar way. $\square$

In the following proposition we determine a set of conditions on the morphism α which are equivalent to the flatness of g.

Proposition 13. *Assume that $(V, \circ, e_V, g)$ is an F-manifold with admissible metric. Let $\{v_1, \cdots, v_r\}$ be a local frame of V, orthonormal with respect to $g_V - \alpha^* g_M$, and $X_1, \cdots, X_n$ a local frame of TM, orthonormal with respect to g_M. Define $\epsilon_i := (g_V - \alpha^* g_M)(v_i, v_i)$ and $\tilde{\epsilon}_j := g(X_j, X_j)$ (thus, $\epsilon_i = \tilde{\epsilon}_j = 1$ in the complex case or when both g_M and $g_V - \alpha^* g_M$ are positive definite). The metric g is flat (and $(V, \circ, e_V, g)$ is Frobenius) if and only if the following conditions hold:*

1) for any vector fields $X, Y, Z, T \in \mathcal{X}(M)$, the expression

$$S_1(X,Y,Z,T) := \sum_{i=1}^{r} \epsilon_i \left(g_M(\nabla^{M,D}_Y(\alpha)(v_i), Z) + g_M(\nabla^{M,D}_Z(\alpha)(v_i), Y) \right) \left(g_M(\nabla^{M,D}_X(\alpha)(v_i), T) + g_M(\nabla^{M}_T(\alpha)(v_i), X) \right)$$

is symmetric in X and Y.

2) for any (local) ∇^M-parallel vector fields $X, Y, Z \in \mathcal{X}(M)$ and D-parallel section $s \in \Gamma(V)$, the expression

$$S_2(X,Y,Z,s) := \sum_{i=1}^{r} \epsilon_i d(\alpha(s)^\flat)(\alpha(v_i), Y) \left(g_M(\nabla^{M,D}_X(\alpha)(v_i), Z) + g_M(\nabla^{M,D}_Z(\alpha)(v_i), X) \right)$$

(where the "$\flat$" denotes the g_M-dual 1-form) satisfies

$$S_2(X,Y,Z,s) - S_2(Y,X,Z,s) = 2\left(g_M(\nabla^M_Y \nabla^M_Z(\alpha(s)), X) - g_M(\nabla^M_X \nabla^M_Z(\alpha(s)), Y) \right).$$

3) for any vector fields $X, Y \in \mathcal{X}(M)$ and (local) D-parallel sections $s, \tilde{s} \in \Gamma(V)$,

$$\sum_{j=1}^{n} \tilde{\epsilon}_j \left(d\alpha(s)^\flat \right)(Y, X_j) \left(d\alpha(\tilde{s})^\flat \right)(X, X_j) + \sum_{i=1}^{r} \epsilon_i \left(d\alpha(s)^\flat \right)(Y, \alpha(v_i)) \left(d\alpha(\tilde{s})^\flat \right)(X, \alpha(v_i)) = 0. \tag{37}$$

Proof. From Lemma 10, relation (33) and (34), for any (local) ∇^M-parallel vector fields $X, Y, Z \in \mathcal{X}(M)$ and D-parallel (local) section $s \in \Gamma(V)$,

$$
\begin{aligned}
(\nabla_{\tilde{X}} \tilde{Y})^{\mathrm{vert}} &= \frac{1}{2} \sum_{i=1}^{r} \epsilon_i \left(g_M(\nabla_X^{M,D}(\alpha)(v_i), Y) + g_M(\nabla_Y^{M,D}(\alpha)(v_i), X) \right) v_i; \\
(\nabla_{\tilde{X}} \tilde{Y})^{\mathrm{hor}} &= -\frac{1}{2} \sum_{i=1}^{r} \epsilon_i \left(g_M(\nabla_X^{M,D}(\alpha)(v_i), Y) + g_M(\nabla_Y^{M,D}(\alpha)(v_i), X) \right) \alpha(v_i); \\
(\nabla_s \tilde{X})^{\mathrm{vert}} &= \frac{1}{2} \sum_{i=1}^{r} \epsilon_i (d\alpha(s)^\flat)(\alpha(v_i), X) v_i; \\
(\nabla_s \bar{X})^{\mathrm{hor}} &= \frac{1}{2} \nabla_X^M (\alpha(s)) - \frac{1}{2} \sum_{j=1}^{n} \tilde{\epsilon}_j g_M(\nabla_{X_j}^M (\alpha(s)), X) X_j \\
&\quad + \frac{1}{2} \sum_{i=1}^{r} \epsilon_i (d\alpha(s)^\flat)(X, \alpha(v_i)) \alpha(v_i).
\end{aligned}
$$

Plugging these relations into the expression of R^∇ from Lemma 53 we readily get our claim. $\square$

After this preliminary material, we can now prove our main result from this section (Theorem 14 below). It turns out that if the base $(M, \circ_M, e_M, g_M)$ is a semisimple Frobenius manifold, the third condition of Proposition 13 simplifies considerably and allows, in the real case, a complete description of all Frobenius structures on V, with positive definite metric, obtained by our method. Recall first that for a semisimple Frobenius manifold the metric is diagonal in canonical coordinates $(u^1, \cdots, u^n)$ and may be written as in (38) below, in terms of a single function η, called the metric potential (to simplify notations, we denote by η_k the derivative $\frac{\partial \eta}{\partial u^k}$; similarly, η_{ij} denotes $\frac{\partial^2 \eta}{\partial u^i \partial u^j}$, etc). For more details about semisimple Frobenius manifolds, see e.g. [9].

Theorem 14. *Let $(M, \circ_M, e_M, g_M)$ be a real semisimple Frobenius manifold with metric*

$$
g_M = \sum_{k=1}^{n} \eta_k du^k \otimes du^k \tag{38}
$$

and non-vanishing rotation coefficients $\gamma_{ij} = \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}}$. Let $V \to M$ be a real vector bundle with a structure of Frobenius algebra $(\circ_V, e_V, g_V)$ along the fibers. Let D be a connection on V and $\alpha : V \to TM$ a morphism such that

$$
\alpha(e_V) = e_M, \quad \alpha(v_1 \circ_V v_2) = \alpha(v_1) \circ_M \alpha(v_2), \quad \forall v_1, v_2 \in V. \tag{39}
$$

Then the almost Frobenius structure $(\circ, e_V, g)$ on V defined by this data (see relations (25) and (31)) is Frobenius with positive definite metric if and only if the following facts hold:

1) the connection D is flat and the Frobenius algebra $(\circ_V, e_V, g_V)$ is D-parallel;

2) the endomorphism α is given by

$$\alpha = \lambda \otimes e_M \tag{40}$$

where $\lambda \in \Gamma(V^*)$ is D-parallel and satisfies

$$\lambda(e_V) = 1, \quad \lambda(s_1 \circ_V s_2) = \lambda(s_1)\lambda(s_2), \quad \forall s_1, s_2 \in \Gamma(V). \tag{41}$$

3) $\eta_k > 0$ for any $1 \leq k \leq n$ and $g_V - (\sum_{k=1}^n \eta_k)\, \lambda \otimes \lambda$ are positive definite.

Proof. Recall, from Propositions 7 and 9 and Remark 8, that $(V, \circ, e_V, g)$ is an F-manifold with admissible metric if and only if D is flat, the Frobenius structure $(\circ_V, e_V, g_V)$ along the fibers is D-parallel and the morphism α maps any (local) D-parallel section s to a vector field which has constant coefficients in a canonical coordinate system of $(M, \circ_M, e_M)$. Moreover, remark that g is positive definite if and only if g_M is positive definite (i.e. $\eta_k > 0$ for any $1 \leq k \leq n$) and $g_V - \alpha^* g_M$ (which is equal to $g_V - (\sum_{k=1}^n \eta_k)\, \lambda \otimes \lambda$ when α given by (40)), is also positive definite. Therefore, it remains to check that the flatness of g determines the morphism α as in (40). For this, we use the third condition of Proposition 13 with $\epsilon_i = \tilde{\epsilon}_j = 1$. For any D-parallel local sections $s, \tilde{s} \in \Gamma(V)$,

$$\alpha(s) = \sum_{k=1}^n a_k \frac{\partial}{\partial u^k}, \quad \alpha(\tilde{s}) = \sum_{k=1}^n \tilde{a}_k \frac{\partial}{\partial u^k} \tag{42}$$

for some constants $a_k, \tilde{a}_k$. In relation (37), let $X := \frac{\partial}{\partial u^p}$ and $Y := \frac{\partial}{\partial u^q}$. It can be checked that with these arguments, relation (37) becomes

$$\begin{aligned} &\sum_{j=1}^n \frac{\eta_{pj}\eta_{qj}}{\eta_j}(a_j - a_q)(\tilde{a}_j - \tilde{a}_p) + \\ &\sum_{i=1}^r \left(\sum_{s=1}^n (a_s - a_q)\eta_{sq} du^s(\alpha(v_i)) \right) \left(\sum_{t=1}^n (\tilde{a}_t - \tilde{a}_p)\eta_{tp} du^t(\alpha(v_i)) \right) = 0. \end{aligned} \tag{43}$$

In (43) let $p = q$ and $s = \tilde{s}$. We obtain:

$$\sum_{j=1}^{n} \frac{\eta_{pj}^2}{\eta_j}(a_j - a_p)^2 + \sum_{i=1}^{r}\left(\sum_{s=1}^{n}(a_s - a_p)\eta_{ps} du^s(\alpha(v_i))\right)^2 = 0.$$

Since η_{pj} are non-vanishing and $\eta_j > 0$ (g_M being positive definite), we deduce that $a_j = a_p$ for any $1 \leq p, j \leq n$, i.e.

$$\alpha(s) = \lambda(s) e_M,$$

where $\lambda(s)$ is a constant. It follows that $\lambda \in \Gamma(V^*)$ is D-parallel. From the algebraic properties (39) of α, we obtain (41). Conversely, if α is of the form (40), with $\lambda \in \Gamma(V^*)$ D-parallel and satisfying (41), then $\nabla^{M,D}(\alpha) = 0$, all three conditions from Proposition 13 hold and the metric g is flat. Our claim follows.

$\square$

5.1 A class of Frobenius manifolds

In this section we study a class of Frobenius structures on the product $M \times \mathbb{K}^r$, where M is Frobenius. It is given by the following theorem.

Theorem 15. *Let $(M, \circ_M, e_M, g_M, E)$ be a Frobenius manifold with Euler field such that $L_E(g_M) = 2g_M$. On the vector space $\mathbb{K}^r$ consider any bilinear, commutative, associative multiplication $\circ_V$, with unit e_V, and $\circ_V$-invariant metric g_V. Let $\lambda \in (\mathbb{K}^r)^*$ such that*

$$\lambda(v \circ_V w) = \lambda(v)\lambda(w), \quad \forall v, w \in \mathbb{K}^r, \quad \lambda(e_V) = 1.$$

Define a multiplication $\circ$ on $T(M \times \mathbb{K}^r)$ by

$$X \circ Y := X \circ_M Y, \quad v \circ w := v \circ_V w \quad X \circ v := \lambda(v) X,$$

where $X, Y \in T_p M$, $v, w \in T_{\vec{\tau}}\mathbb{K}^r = \mathbb{K}^r$, and a metric g on $M \times \mathbb{K}^r$ by

$$g(X,Y) = g_M(X,Y), \quad g(v,w) = g_V(v,w), \quad g(X,v) = \lambda(v) g_M(X, e_M). \tag{44}$$

If the bilinear form

$$g_{M,V} := g_V - g_M(e_M, e_M)\lambda \otimes \lambda \tag{45}$$

on $\mathbb{C}^r$ is non-degenerate, then $(M \times \mathbb{K}^r, \circ, e_V, g)$ is a Frobenius manifold, with Euler field $E + R$ (where $R_{(p,\vec{\tau})} = \sum_{k=1}^{r} \tau_k \frac{\partial}{\partial \tau_k}$ is the radial field), and $L_{E+R}(g) = 2g$.

Proof. The claim that $(M \times \mathbb{K}^r, \circ, e_V, g)$ is Frobenius is straightforward, from Propositions 7, 9 and 13, with

$$\pi : V = M \times \mathbb{K}^r \to M, \quad \pi(p, \tau_1, \cdots, \tau_r) = p$$

the trivial bundle, D the trivial standard connection and $\alpha : V \to TM$ given by $\alpha(v) = \lambda(v) e_M$, for any $v \in V_{\pi(v)} \cong \mathbb{K}^r$. The statement involving the Euler fields can be checked directly. $\square$

Remark 16. We remark that the Frobenius structure from the above theorem is more general than those provided by iterations of adding a variable to a Frobenius manifold (see Proposition 6). The reason is that the multiplication $\circ_V$ on $\mathbb{K}^r$ from the above theorem can be chosen arbitrarily (as long as it is commutative, associative, with unit), while in Proposition 6 the multiplication $\circ^r$ restricted to vertical vectors $\frac{\partial}{\partial \tau_i}$ has a very precise form (in particular, it is semisimple, with canonical basis $\{w_1 := \frac{\partial}{\partial \tau_1}, w_2 := \frac{\partial}{\partial \tau_2} - \frac{\partial}{\partial \tau_1}, \cdots, w_r := \frac{\partial}{\partial \tau_r} - \frac{\partial}{\partial \tau_{r-1}}\}$).

Remark 17. It is easy to relate the potential F_M of the Frobenius structure on M to the potential F of the Frobenius structure on $M \times \mathbb{K}^r$, from Theorem 15. If (t^i) are flat coordinates for the metric g_M, then (t^i, τ^j) (with (τ^j) the standard coordinate system of $\mathbb{K}^r$) are flat coordinates for g and one may check that

$$F(t^i, \tau^j) = F_M(t^i) + \frac{1}{2}\left(\sum_{s=1}^{r} \lambda_s \tau_s\right)\left(\sum_{i,j=1}^{n} g_{ij} t^i t^j\right) + \frac{1}{2}\left(\sum_{s,k=1}^{r} \lambda_{sk} \tau^s \tau^k\right)\left(\sum_{j=1}^{n} \epsilon_j t^j\right) + \frac{1}{6}\sum_{s,k,j} g_{skj} \tau^s \tau^k \tau^j$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right), \quad \lambda_s := \lambda\left(\frac{\partial}{\partial \tau^s}\right), \quad \lambda_{sk} := \lambda\left(\frac{\partial}{\partial \tau^s} \circ_V \frac{\partial}{\partial \tau^k}\right),$$
$$g_{skj} = g\left(\frac{\partial}{\partial \tau^s} \circ_V \frac{\partial}{\partial \tau^k}, \frac{\partial}{\partial \tau^j}\right), \quad \epsilon_j = g_M\left(e_M, \frac{\partial}{\partial t^j}\right)$$

are all constants.

Now we show how the Frobenius structure on $M \times \mathbb{K}^r$ from Theorem 15 can be obtained from a Saito bundle. We denote by ϕ^M and ϕ^V the Higgs fields determined by $\circ_M$ and $\circ_V$, i.e.

$$\phi^M_X(Y) = -X \circ_M Y, \quad \phi^V_v(w) = -v \circ_V w, \quad \forall X, Y \in TM, \quad \forall v, w \in \mathbb{K}^r,$$

considered as 1-forms on M with values in $\mathrm{End}(TM \oplus V)$ (acting trivially on V and TM, respectively). Using ϕ^M and ϕ^V we define an $\mathrm{End}(\pi^*(TM \oplus V))$-valued 1-form $\phi^{M,V}$ on $M \times \mathbb{K}^r$, which, at $(p, \vec{\tau}) \in M \times \mathbb{K}^r$, is given by

$$\begin{aligned} &\phi_X^{M,V}(\pi^* Y) = \pi^* \phi_X^M(Y), \quad \phi_X^{M,V}|_{\pi^* V} = 0 \\ &\phi_v^{M,V}(\pi^* w) = \pi^* \phi_v^V(w), \quad \phi_v^{M,V}|_{\pi^* TM} = -\lambda(v)\mathrm{Id}_{\pi^* TM}, \end{aligned}$$

for any $X, Y \in T_pM$ and $v, w \in T_{\vec{\tau}}\mathbb{K}^r = \mathbb{K}^r = V_p$.

Proposition 18. *On the bundle $\pi^*(TM \oplus V) \to M \times \mathbb{K}^r$ consider the following data:*

1) the Higgs field $\phi^{M,V}$;

2) the connection $\pi^*(\nabla^M \oplus D)$, where ∇^M is the Levi-Civita connection of g_M and D is the standard trivial connection on the bundle V;

3) the metric $\pi^*(g_M \oplus g_{M,V})$, where $g_{M,V}$, defined by (45), is considered as a (constant) metric on the bundle V;

4) the endomorphism $R_0' \in \mathrm{End}\,(\pi^*(TM \oplus V))$, which, at a point $(p, \vec{\tau}) \in M \times \mathbb{K}^r$, is defined by

$$\begin{aligned} (R_0')_{(p,\vec{\tau})}(\pi^* X) &:= \pi^*(X \circ_M E^M) + \lambda(\vec{\tau})\pi^* X, \\ (R_0')_{(p,\vec{\tau})}(\pi^* v) &:= \pi^*(v \circ_V \vec{\tau}), \end{aligned}$$

for any $X \in T_pM$ and $v \in V_p = \mathbb{K}^r$.

5) the endomorphism $R_\infty' \in \mathrm{End}\,(\pi^*(TM \oplus V))$ defined by

$$R_\infty'|_{\pi^* TM} := \pi^*(\nabla^M E - \mathrm{Id}_{TM}), \quad R_\infty'|_{\pi^* V} = 0.$$

Then $\pi^*(TM \oplus V)$ with this data is a Saito bundle of weight zero. The Frobenius structure from Theorem 15 can be obtained from this Saito bundle, by choosing $\pi^*(e_M + e_V)$ as a primitive homogeneus section.

Proof. The proof is straightforward and we omit the details. We only remark that $\pi^*(e_M + e_V)$ is $\pi^*(\nabla^M \oplus D)$-parallel, belongs to the kernel of R_∞' (hence is homogeneous), and the map

$$\psi : T(M \times \mathbb{K}^r) \to \pi^*(TM \oplus V), \quad Z \to \psi(Z) := -\phi_Z\left(\pi^*(e_M + e_V)\right) \tag{46}$$

is given by

$$\psi_{(p,\vec{\tau})}(X) = \pi^*(X), \quad \psi_{(p,\vec{\tau})}(v) = \pi^*(\lambda(v)e_M + v), \tag{47}$$

for any $X \in T_pM$ and $v \in T_{\vec{\tau}}\mathbb{K}^r = \mathbb{K}^r = V_p$, and is an isomorphism.

$\square$

Remark 19. In the setting of Proposition 18, assume that $V = L$ is trivial of rank one. Changing the trivialization of V if necessary, we may assume that e_V (viewed as a vector field on V), is $\frac{\partial}{\partial \tau}$. Then $\lambda = d\tau$ and $\alpha = d\tau \otimes e_M$. Let g_V be defined by $g_V(e_V, e_V) = 1 + g_M(e_M, e_M)$. The resulting Saito structure from Proposition 18 coincides with the Saito structure from Remark 5, the latter with $d = 2$, $w = 0$, $v = e_V$ and $\tilde{g} := g_V - g_M(e_M, e_M)\lambda \otimes \lambda$. Therefore, the above proposition is a generalization of adding a variable to a Frobenius manifold, using abstract Saito bundles.

6 tt^*-geometry on V

The tt^*-equations may be defined on any holomorphic vector bundle (with a pseudo-Hermitian metric and holomorphic Higgs field), but in our considerations we shall only consider them on the holomorphic tangent bundle of a complex manifold.

Notations 20. In this section we use slightly different notations from those employed until now. The reason is that the tt^*-equations involve vector fields of type $(1,0)$ and $(0,1)$ as well. In order to avoid confusion, the holomorphic tangent bundle of a complex manifold M will be denoted by $T^{1,0}M$ (rather than TM, how it was denoted before). The sheaf of smooth (respectively holomorphic) vector fields will be denoted by $\mathcal{T}^{1,0}_M$ (respectively, $\mathcal{T}_M$). For an holomorphic vector bundle $V \to M$, $\Gamma(V)$ will denote, as before, the sheaf of holomorphic sections of V.

We begin by recalling the tt^*-equations.

6.1 The tt^*-equations

Given a complex manifold M together with an (associative, commutative, with unit) multiplication $\circ$ on $T^{1,0}M$ and a pseudo-Hermitian metric h, the tt^*-equations are the followings:

$$(\partial^D \phi)_{Z_1, Z_2} = 0, \quad R^D_{Z_1, \bar{Z}_2} + [\phi_{Z_1}, \phi^\dagger_{\bar{Z}_2}] = 0, \quad Z_1, Z_2 \in \mathcal{T}^{1,0}M$$

where

$$(\partial^D \phi)_{Z_1, Z_2} := D_{Z_1}(\phi_{Z_2}) - D_{Z_2}(\phi_{Z_1}) - \phi_{[Z_1, Z_2]}.$$

Above $\phi \in \Omega^{1,0}(M, \mathrm{End}T^{1,0}M)$ is the Higgs field determined by the multiplication, i.e. $\phi_X(Y) := -X \circ Y$, $\phi^\dagger \in \Omega^{0,1}(M, \mathrm{End}T^{1,0}M)$ is the h-adjoint

of ϕ and D is the Chern connection of h. In our considerations, the pseudo-Hermitian metric will be given by $h(Z_1, Z_2) = g(Z_1, kZ_2)$, where g is a (multiplication) invariant metric on $T^{1,0}M$ and $k : T^{1,0}M \to T^{1,0}M$ is a real structure, compatible with g, i.e. $g(kZ_1, kZ_2) = \overline{g(Z_1, Z_2)}$, for any $Z_1, Z_2 \in T^{1,0}M$. (Recall that a real structure on a vector bundle is a fiber-preserving complex anti-linear involution). In this case, $\phi^{\dagger}_{\bar{Z}} = k\phi_Z k$, for any $Z \in T^{1,0}M$. The compatibility condition insures that h defined as above in terms of g and k is indeed a pseudo-Hermitian metric.

Example 21. On any semisimple complex Frobenius manifold $(M, \circ, e, g)$ one can define a diagonal real structure, such that the tt^*-equations hold (see Theorem 2.4 of [8]). More precisely, let η be the metric potential in canonical coordinates $(u^1, \cdots, u^n)$, i.e. $g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \delta_{ij}\eta_i$, for any i, j (with $\eta_i = \frac{\partial \eta}{\partial u^i}$). Define a real structure on $T^{1,0}M$, compatible with g, by $k\left(\frac{\partial}{\partial u^i}\right) = \frac{|\eta_i|}{\eta_i}\frac{\partial}{\partial u^i}$, for any i. The Chern connection D of the associated pseudo-Hermitian metric $h = g(\cdot, k\cdot)$ is given by

$$D_Z\left(\frac{\partial}{\partial u^i}\right) = Z\left(\log|\eta_i|\right)\frac{\partial}{\partial u^i}, \quad Z \in \mathcal{T}^{1,0}_M, \quad 1 \leq i \leq n \tag{48}$$

and one may show that the tt^*-equations hold. Remark that in this example the second tt^*-equation decouples: D is flat and the Higgs field commutes with its h-adjoint. Other examples of tt^*-structures may be constructed on the base space of the universal unfolding of various Laurent polynomials [11]. There is also a real version of the tt^*-equations, which relate harmonic Higgs bundles with special geometries, see e.g. [1, 15].

6.2 The main result

Let $\pi : V \to M$ be an holomorphic vector bundle whose base has an holomorphic Frobenius structure $(\circ_M, e_M, g_M)$, the typical fiber has a Frobenius algebra $(\circ_V, e_V, g_V)$, and which comes equipped with an holomorphic connection D and an holomorphic morphism $\alpha : V \to T^{1,0}M$, preserving the multiplications and the unit fields. Let $\circ$ and g be the induced multiplication and metric on the manifold V, defined as usual by (25) and (31). We denote by ϕ, ϕ^M and ϕ^V the Higgs fields associated to $\circ$, $\circ_M$ and $\circ_V$ respectively.

Assumption 22. In the following considerations, we assume that $(V, \circ, e_V)$ is an F-manifold, i.e. all conditions from Proposition 7 are satisfied - in particular, D is flat. (This assumption is not restrictive: our goal is to construct solutions to the tt^*-equations on $(V, \circ, e_V, g)$; but, as shown in Lemma 4.3 of [5], the first tt^*-equation implies the F-manifold condition).

Let k_M be a real structure on $T^{1,0}M$, compatible with g_M and $h_M := g_M(\cdot, k_M\cdot)$ the associated pseudo-Hermitian metric. Similarly, let k_V be a real structure on the bundle V, compatible with g_V, and $h_V := g_V(\cdot,\cdot)$. Using k_M and k_V we construct a real structure k on $T^{1,0}V$, which on the D-horizontal subbundle of $T^{1,0}V$ coincides with k_M and on the vertical subbundle coincides with k_V. We assume that

$$\alpha \circ k_V = k_M \circ \alpha \tag{49}$$

which means that k is compatible with g (easy check). Let $h := g(\cdot, k\cdot)$, given by

$$h(\tilde{X},\tilde{Y}) = h_M(X,Y), \quad h(\tilde{X}, v) = h_M(X,\alpha(v))$$
$$h(v_1,v_2) = h_V(v_1,v_2), \quad h(v,\tilde{X}) = h_M(\alpha(v),X),$$

for any $X, Y \in T^{1,0}_p M$ and $v, v_1, v_2 \in V_p$ ($p \in M$). Our main result is the following.

Theorem 23. *In the above setting, the tt*-equations hold on $(V,\circ,h)$ if and only if they hold on $(M,\circ_M,h_M)$ and the following additional conditions are true:*

i) for any local D-parallel section $s \in \Gamma(V)$,

$$D^M_X(\alpha(s)\circ_M Z) = \alpha(s)\circ_M D^M_X(Z), \quad \forall X, Z \in \mathcal{T}_M, \tag{50}$$

where D^M is the Chern connection of h_M;

ii) The $(1,0)$-part of the Chern connection D^V of h_V is related to D by

$$D^V_X s = D_X s + (D^V_X e_V)\circ_V s, \quad \forall X \in \mathcal{T}_M, \quad \forall s \in \Gamma(V); \tag{51}$$

iii) for any $X \in \mathcal{T}_M$ and $s, s_1, s_2 \in \Gamma(V)$,

$$[\phi^M_X, k_M \phi^M_{\alpha(s)} k_M] = 0, \quad [\phi^V_{s_1}, k_V \phi^V_{s_2} k_V] = 0; \tag{52}$$

iv) For any $X, Y \in \mathcal{T}_M$ and $s, s_1 \in \Gamma(V)$,

$$\begin{aligned} &h_V\left(R^{D^V}_{X,\bar{Y}} s, s_1\right) - h_M\left(R^{D^M}_{X,\bar{Y}}\alpha(s),\alpha(s_1)\right) = \\ &h_M\left(\mathcal{D}_X(\alpha)(s), D^M_Y(\alpha(s_1)) - \alpha(D^{M,V}_Y s_1)\right), \end{aligned} \tag{53}$$

where $D^{M,V}$ is the Chern connection of $h_V - \alpha^* h_M$ and

$$\mathcal{D}_X(\alpha)(s) := D^M_X(\alpha(s)) - \alpha(D^V_X s).$$

We divide the proof of the above theorem in two stages. In a first stage, we find the conditions for the first tt^*-equation to hold on $(V, \circ, h)$ (see Proposition 24). In a second stage, we compute the curvature of the Chern connection of h (see Lemma 25). This will readily imply the conditions for the second tt^*-equation to hold on $(V, \circ, h)$ and will conclude the proof of Theorem 23. Details are as follows.

Proposition 24. *The first tt^*-equation holds on $(V, \circ, h)$ if and only if it holds on $(M, \circ_M, h_M)$ and conditions (50) and (51) are true.*

Proof. Let D^c be the Chern connection of h. Using the relation

$$\mathcal{W}_1 h(\mathcal{W}_2, \mathcal{W}_3) = h\left(D^c_{\mathcal{W}_1}(\mathcal{W}_2), \mathcal{W}_3\right), \quad \forall \mathcal{W}_i \in \mathcal{T}_V$$

we obtain the expression of D^c as follows:

$$\begin{aligned} &D^c_{\tilde{X}}\tilde{Y} = \left[D^M_X Y\right]^{\sim}, \quad h\left(D^c_{\tilde{X}}s, \tilde{Y}\right) = h_M\left(D^M_X(\alpha(s)), Y\right)\\ &D^c_s\tilde{Y} = D^c_{s_1}s_2 = 0, \quad h\left(D^c_{\tilde{X}}s, s_1\right) = h_V\left(D^V_X s, s_1\right), \end{aligned} \tag{54}$$

for any $s, s_1, s_2 \in \Gamma(V)$ and $X, Y \in \mathcal{T}_M$. Using these relations, together with

$$(\partial^{D^c}\phi)_{\mathcal{W}_1,\mathcal{W}_2} = D^c_{\mathcal{W}_1}(\phi_{\mathcal{W}_2}) - D^c_{\mathcal{W}_2}(\phi_{\mathcal{W}_1}) - \phi_{[\mathcal{W}_1,\mathcal{W}_2]}$$

and the flatness of D, we obtain

$$\begin{aligned} (\partial^{D^c}\phi)_{\tilde{X},\tilde{Y}}(\tilde{Z}) &= \left[(\partial^{D^M}\phi^M)_{X,Y}(Z)\right]^{\sim}\\ (\partial^{D^c}\phi)_{\tilde{X},\tilde{Y}}(s) &= \left[(\partial^{D^M}\phi^M_{\alpha(s)})_{X,Y}\right]^{\sim} - \tilde{X}\circ D^c_{\tilde{Y}}s + \tilde{Y}\circ D^c_{\tilde{X}}s\\ (\partial^{D^c}\phi)_{s_1,s_2}(\tilde{Z}) &= (\partial^{D^c}\phi)_{s_1,s_2}(s) = 0 \end{aligned}$$

for any $X, Y, Z \in \mathcal{T}_M$, $s, s_1 \in \Gamma(V)$, where

$$\left(\partial^{D^M}\phi^M_{\alpha(s)}\right)_{X,Y} := -D^M_X\left(\alpha(s)\circ_M Y\right) + D^M_Y\left(\alpha(s)\circ_M X\right) + \alpha(s)\circ_M [X, Y].$$

If, moreover, s_1 is D-parallel, then

$$\begin{aligned} (\partial^{D^c}\phi)_{s_1,\tilde{X}}(\tilde{Z}) &= \left[D^M_X\left(\alpha(s_1)\circ_M Z\right) - \alpha(s_1)\circ_M D^M_X(Z)\right]^{\sim}\\ (\partial^{D^c}\phi)_{s_1,\tilde{X}}(s) &= D^c_{\tilde{X}}(s\circ_V s_1) - s_1\circ D^c_{\tilde{X}}s. \end{aligned}$$

It follows that $\partial^{D^c}\phi = 0$ if and only if $\partial^{D^M}\phi^M = 0$, relation (50) holds and the following two relations hold as well:

$$\begin{aligned} & D^c_{\tilde{X}}(s\circ_V s_1) - s_1\circ D^c_{\tilde{X}} s = 0\\ & \left[(\partial^{D^M}\phi^M_{\alpha(s)})_{X,Y}\right]^{\sim} - \tilde{X}\circ D^c_{\tilde{Y}} s + \tilde{Y}\circ D^c_{\tilde{X}} s = 0, \end{aligned} \tag{55}$$

for any $X, Y\in \mathcal{T}_M$, $s, s_1\in\Gamma(V)$ and $D(s_1) = 0$. In the remaining part of the proof we assume that $\partial^{D^M}\phi^M = 0$ and that relation (50) is true and we show that relations (55) are equivalent to (51). We begin with the first relation (55). Notice that: for any $s_2\in\Gamma(V)$,

$$\begin{aligned} h\left(s_1\circ D^c_{\tilde{X}}s, s_2\right) &= g\left(D^c_{\tilde{X}}s, s_1\circ k_V(s_2)\right) = h\left(D^c_{\tilde{X}}s, k_V(s_1\circ_V k_V(s_2))\right)\\ &= h_V\left(D^V_X s, k_V\left(s_1\circ_V k_V(s_2)\right)\right) = g_V\left(D^V_X s, s_1\circ_V k_V(s_2)\right)\\ &= h_V\left(s_1\circ_V D^V_X s, s_2\right), \end{aligned}\tag{56}$$

where we used (54). Also from (54),

$$h\left(D^c_{\tilde{X}}(s\circ s_1), s_2\right) = h_V\left(D^V_X\left(s\circ_V s_1\right), s_2\right). \tag{57}$$

Combining (56) and (57) we obtain

$$h\left(D^c_{\tilde{X}}(s\circ_V s_1) - s_1\circ D^c_{\tilde{X}}s, s_2\right) = h_V\left(D^V_X(s\circ_V s_1) - s_1\circ_V D^V_X s, s_2\right). \tag{58}$$

A similar computation shows that

$$h\left(D^c_{\tilde{X}}(s\circ_V s_1) - s_1\circ D^c_{\tilde{X}}s, \tilde{Z}\right) = h_M\left(D^M_X\left(\alpha(s\circ_V s_1)\right) - \alpha(s_1)\circ_M D^M_X(\alpha(s)), Z\right),$$

which follows from (50). We proved that the first relation (55) gives

$$D^V_X(s\circ_V s_1) = (D^V_X s)\circ_V s_1, \tag{59}$$

for any $X\in\mathcal{T}_M$, $s, s_1\in\Gamma(V)$ with $D(s_1) = 0$, and this is equivalent to (51) (easy check). It remains to consider the second relation (55). For this, one shows that

$$\begin{aligned} h\left(\tilde{X}\circ D^c_{\tilde{Y}}s, s_2\right) &= h_M\left(X\circ_M D^M_Y\left(\alpha(s)\right), \alpha(s_2)\right)\\ h\left(\tilde{X}\circ D^c_{\tilde{Y}}s, \tilde{Z}\right) &= h_M\left(X\circ_M D^M_Y\left(\alpha(s)\right), Z\right) \end{aligned}$$

which imply

$$
\begin{aligned}
& h\left(\left[(\partial^{D^M}\phi^M_{\alpha(s)})_{X,Y}\right]^{\sim} - \tilde{X}\circ D^c_{\tilde{Y}}s + \tilde{Y}\circ D^c_{\tilde{X}}s, s_2\right) \\
& = h_M\left((\partial^{D^M}\phi^M)_{X,Y}(\alpha(s)), \alpha(s_2)\right) \\
& h\left(\left[(\partial^{D^M}\phi^M_{\alpha(s)})_{X,Y}\right]^{\sim} - \tilde{X}\circ D^c_{\tilde{Y}}s + \tilde{Y}\circ D^c_{\tilde{X}}s, Z\right) \\
& = h_M\left((\partial^{D^M}\phi^M)_{X,Y}(\alpha(s)), Z\right),
\end{aligned}
$$

for any $X, Y \in \mathcal{T}^{1,0}_M$ and $s, s_1, s_2 \in \Gamma(V)$, with $D(s_1) = 0$. Thus, the second relation (55) is a consequence of the first tt^*-equation $\partial^{D^M}\phi^M = 0$. Our claim follows.

□

Lemma 25. *The curvature of the Chern connection D^c of h has the following expression: for any $X, Y, Z \in \mathcal{T}_M$ and $s, s_1, s_2 \in \Gamma(V)$,*

$$
\begin{aligned}
& R^{D^c}_{\tilde{X},\bar{\tilde{Y}}}\tilde{Z} = \left[R^{D^M}_{X,\bar{Y}}Z\right]^{\sim}, \quad h\left(R^{D^c}_{\tilde{X},\bar{\tilde{Y}}}s, \tilde{Z}\right) = h_M\left(R^{D^M}_{X,\bar{Y}}\alpha(s), Z\right), \\
& R^{D^c}_{s,\bar{\tilde{X}}}\tilde{Y} = R^{D^c}_{\tilde{X},\bar{s}}\tilde{Y} = R^{D^c}_{s_1,\bar{s}_2}\tilde{Y} = R^{D^c}_{s_1,\bar{\tilde{X}}}s = R^{D^c}_{\tilde{X},\bar{s}_1}s = R^{D^c}_{s_1,\bar{s}_2}s = 0
\end{aligned}
$$

and

$$
h\left(R^{D^c}_{\tilde{X},\bar{\tilde{Y}}}s, s_1\right) = h_V\left(R^{D^V}_{X,\bar{Y}}s, s_1\right) + h_M\left(\mathcal{D}_X(\alpha)(s), D^M_Y(\alpha(s_1)) - \alpha(D^{M,V}_Y s_1)\right). \tag{60}
$$

Proof. We only show (60), the other components of R^{D^c} can be obtained by straightforward computations, using (54). To prove (60), one first notices, from

$$
h\left(D^c_{\tilde{X}}s, \tilde{Z}\right) = h_M\left(D^M_X(\alpha(s)), Z\right), \quad h\left(D^c_{\tilde{X}}s, s_1\right) = h_V\left(D^V_X s, s_1\right),
$$

that

$$
D^c_{\tilde{X}}s = s_0 + \tilde{X}_0
$$

where s_0 is a section of V (not necessarily holomorphic), determined by

$$
(h_V - \alpha^* h_M)(s_0, s_1) = h_V\left(D^V_X s, s_1\right) - h_M\left(D^M_X(\alpha(s)), \alpha(s_1)\right), \tag{61}
$$

for any $s_1 \in \Gamma(V)$, and $X_0 \in \mathcal{T}^{1,0}_M$ is determined by

$$
h_M(X_0, Z) = h_M(D^M_X(\alpha(s)), Z) - h_M(\alpha(s_0), Z), \tag{62}
$$

for any $Z \in \mathcal{T}_M$. Now,

$$h\left(R^{D^c}_{\tilde{X},\bar{\tilde{Y}}} s, s_1\right) = -h\left(\bar{\partial}_{\bar{\tilde{Y}}} D^c_{\tilde{X}} s, s_1\right) = -h_V\left(\bar{\partial}_{\bar{Y}} s_0, s_1\right) - h_M\left(\bar{\partial}_{\bar{Y}} X_0, \alpha(s_1)\right), \tag{63}$$

where in the first equality we used that $X, Y \in \mathcal{T}_M$ and the flatness of D (hence $[\tilde{X}, \bar{\tilde{Y}}] = 0$), and also that s is holomorphic (hence $\bar{\partial}_{\bar{Y}} s = 0$). Therefore, we need to compute $\bar{\partial}_{\bar{Y}} s_0$ and $\bar{\partial}_{\bar{Y}} X_0$. By applying $\bar{\partial}_{\bar{Y}}$ to (61), it is easy to see that

$$\begin{aligned}(h_V - \alpha^* h_M)\left(\bar{\partial}_{\bar{Y}} s_0, s_1\right) &= h_M\left(R^{D^M}_{X,\bar{Y}} \alpha(s), \alpha(s_1)\right) - h_V\left(R^{D^V}_{X,\bar{Y}} s, s_1\right)\\ &+ h_M\left(D^M_X(\alpha(s)), \alpha(D^{M,V}_Y s_1) - D^M_Y(\alpha(s_1))\right)\\ &+ h_V\left(D^V_X s, D^V_Y s_1 - D^{M,V}_Y s_1\right). \end{aligned}\tag{64}$$

Similarly, one computes,

$$h_M\left(\bar{\partial}_{\bar{Y}} X_0, Z\right) = -h_M\left(R^{D^M}_{X,\bar{Y}} \alpha(s), Z\right) - h_M\left(\alpha(\bar{\partial}_{\bar{Y}} s_0), Z\right). \tag{65}$$

Combining this relation with (63), we obtain

$$\begin{aligned}h\left(R^{D^c}_{\tilde{X},\bar{\tilde{Y}}} s, s_1\right) &= -h_V\left(\bar{\partial}_{\bar{Y}} s_0, s_1\right) - h_M\left(\bar{\partial}_{\bar{Y}} X_0, \alpha(s_1)\right)\\ &= (-h_V + \alpha^* h_M)\left(\bar{\partial}_{\bar{Y}} s_0, s_1\right) + h_M\left(R^{D^M}_{X,\bar{Y}} \alpha(s), \alpha(s_1)\right),\end{aligned}$$

or, using (64),

$$\begin{aligned}h\left(R^{D^c}_{\tilde{X},\bar{\tilde{Y}}} s, s_1\right) &= -h_V\left(D^V_X s, D^V_Y s_1\right) + h_M\left(D^M_X(\alpha(s)), D^M_Y(\alpha(s_1))\right)\\ &+ h_V\left(D^V_X s, D^{M,V}_Y s_1\right) - h_M\left(D^M_X(\alpha(s)), \alpha(D^{M,V}_Y s_1)\right)\\ &+ h_V\left(R^{D^V}_{X,\bar{Y}} s, s_1\right).\end{aligned}\tag{66}$$

Denote by E_1 and E_2 the term on the first line, respectively the second line, on the right hand side of (66). Remark that

$$\begin{aligned}E_1 = &- \bar{Y} h_V\left(D^V_X s, s_1\right) + h_V\left(\bar{\partial}_{\bar{Y}} D^V_X s, s_1\right) + \bar{Y} h_M\left(D^M_X(\alpha(s)), \alpha(s_1)\right)\\ &- h_M\left(\bar{\partial}_{\bar{Y}} D^M_X(\alpha(s)), \alpha(s_1)\right)\\ &= -\bar{Y} h_V\left(D^V_X s, s_1\right) - h_V\left(R^{D^V}_{X,\bar{Y}}(s), s_1\right) + \bar{Y} h_M\left(D^M_X(\alpha(s)), \alpha(s_1)\right)\\ &+ h_M\left(R^{D^M}_{X,\bar{Y}} \alpha(s), \alpha(s_1)\right).\end{aligned}$$

A similar computation shows that

$$\begin{aligned}
E_2 &= (h_V - \alpha^* h_M)\left(D^V_X s, D^{M,V}_Y s_1\right) - h_M\left(\mathcal{D}_X(\alpha)(s), \alpha(D^{M,V}_Y s_1)\right)\\
&= \bar{Y} h_V\left(D^V_X s, s_1\right) - \bar{Y} h_M\left(\alpha(D^V_X s), \alpha(s_1)\right) + (h_V - \alpha^* h_M)\left(R^{D^V}_{X,\bar{Y}} s, s_1\right)\\
&\quad - h_M\left(\mathcal{D}_X(\alpha)(s), \alpha(D^{M,V}_Y s_1)\right)
\end{aligned}$$

and we obtain

$$E_1 + E_2 = h_M\left(\mathcal{D}_X(\alpha)(s), D^M_Y(\alpha(s_1)) - \alpha(D^{M,V}_Y s_1)\right). \tag{67}$$

Combining (66) with (67) we obtain (60), as required. $\square$

Theorem 23 follows from Proposition 24, Lemma 25 and the following brackets:

$$\begin{aligned}
&[\phi_{\tilde{X}}, k\phi_{\tilde{Y}}k](\tilde{Z}) = [\phi^M_X, k_M\phi^M_Y k_M](Z)\tilde{}, \ [\phi_{\tilde{X}}, k\phi_{\tilde{Y}}k](s) = [\phi^M_X, k_M\phi^M_Y k_M](\alpha(s))\tilde{}\\
&[\phi_{\tilde{X}}, k\phi_s k](\tilde{Z}) = [\phi^M_X, k_M\phi^M_{\alpha(s)} k_M](Z)\tilde{}, \ [\phi_{\tilde{X}}, k\phi_s k](s_1) = [\phi^M_X, k_M\phi^M_{\alpha(s)} k_M](\alpha(s_1))\tilde{}\\
&[\phi_s, k\phi_{\tilde{X}}k](\tilde{Z}) = [\phi^M_{\alpha(s)}, k_M\phi^M_X k_M](Z)\tilde{}, \ [\phi_s, k\phi_{\tilde{X}}k](s_1) = [\phi^M_{\alpha(s)}, k_M\phi^M_X k_M]\left(\alpha(s_1)\right)\tilde{}\\
&[\phi_{s_1}, k\phi_{s_2}k](\tilde{Z}) = [\phi^M_{\alpha(s_1)}, k_M\phi^M_{\alpha(s_2)} k_M](Z)\tilde{}, \ [\phi_{s_1}, k\phi_{s_2}k](s_3) = [\phi^V_{s_1}, k_V\phi^V_{s_2} k_V](s_3)
\end{aligned}$$

Remark 26. In the setting of Theorem 23, it may be checked that the Chern connection D^c of h preserves g, i.e. $D^c(g) = 0$, if and only if $D^M(g_M) = 0$ and $D^V(g_V) = 0$.

Example 27. We now construct an example where all conditions from Proposition 7 and Theorem 23 are satisfied and hence the tt^*-equations hold on V. Consider a complex semisimple Frobenius manifold $(M, \circ_M, e_M, g_M)$ with metric potential η in canonical coordinates $(u^1, \cdots, u^n)$. Define a diagonal real structure k_M on $T^{1,0}M$, like in Example 21. Let $V \to M$ be a rank n-holomorphic vector bundle and assume there is an (holomorphic) bundle isomorphism $\alpha : V \to T^{1,0}M$. Identifying V with $T^{1,0}M$ using α, we obtain a multiplication $\circ_V$ and a real structure k_V on V, induced by $\circ_M$ and k_M respectively. Let $g_V := k_0\alpha^* g_M$, where $k_0 \in \mathbb{R} \setminus \{1\}$ is fixed. Note that $g_V - \alpha^* g_M = (k_0 - 1)\alpha^* g_M$ is non-degenerate and g_V is compatible with k_V. It remains to define the connection D. It is determined by the condition that an (holomorphic) section s of V is D-parallel if and only if $\alpha(s)$ has constant coefficients in the coordinate system $(u^1, \cdots, u^n)$. In this setting, we claim that all conditions from Proposition 7 and Theorem 23 are satisfied. This may be checked easily. For example, relation (50) follows from

$$D^M_X\left(\frac{\partial}{\partial u^i} \circ_M Z\right) = \frac{\partial}{\partial u^i} \circ_M D^M_X(Z), \quad \forall X, Z \in \mathcal{T}_M$$

which, in turn, is a consequence of (48). Similarly, to prove (53) one remarks that the pseudo-Hermitian metric $h_V := g_V(\cdot, k_V \cdot)$ is given by $h_V = k_0 \alpha^* h_M$ and its Chern connection D^V is related to D^M by

$$D^V_X s = \alpha^{-1} D^M_X (\alpha(s)), \quad \forall s \in \Gamma(V), \quad \forall X \in \mathcal{T}_M \tag{68}$$

and is flat (because D^M is flat, see Example 21). From (68), $\mathcal{D}(\alpha) = 0$ and relation (53) follows. In a similar way one may check the other conditions from Proposition 7 and Theorem 23 and hence the tt^*-equations hold on V. Finally, remark that $D(g_V) \neq 0$ (unless η_i are constant) and from Proposition 9, g is not an admissible metric on the F-manifold $(V, \circ, e_V)$. From Remark 26, the Chern connection of h preserves g (because $D^M(g_M) = 0$ and $D^V(g_V) = 0$.

It is natural to ask if the Frobenius manifold $(V = M \times \mathbb{C}^r, \circ, e_V, g)$ from Theorem 15 (with $\mathbb{K} = \mathbb{C}$) may be given a real structure k in the framework of this section, such that the tt^*-equations hold. It turns out that imposing the tt^*-equations in this setting is a very strong condition, due to the very special form of the morphism α from (the proof of) Theorem 15. More precisely, remark that if k_M is a real structure on M (compatible with g_M) and k_V is a real structure on the trivial bundle $V = M \times \mathbb{C}^r \rightarrow M$ (compatible with g_V), such that $\alpha \circ k_V = k_M \circ \alpha$, with $\alpha = \lambda \otimes e_M$ as in the proof of Theorem 15, then $k_M(e_M)$ is a constant multiple of e_M. However, this fact, together with the tt^*-equations on M (which, according to Theorem 23, is a necessary condition for the tt^*-equations to hold on V) impose strong restrictions on the base Frobenius manifold $(M, \circ_M, e_M, g_M)$, as shown in the following proposition (the condition $D(g) = 0$ below holds for a large class of tt^*-structures, e.g. CV-structures or harmonic Frobenius structures [5, 11]).

Proposition 28. *Assume that the tt^*-equations hold on a complex Frobenius manifold $(M, \circ, e, g)$ with real structure k, compatible with g, and $k(e) = \mu e$, where μ is a constant. Assume, moreover, that the Chern connection D of $h = g(\cdot, k \cdot)$ preserves g. Then the Frobenius manifold $(M, \circ, e, g)$ is trivial, k is constant in flat coordinates for g and satisfies*

$$k(Z_1 \circ Z_2) = \bar{\mu} k(Z_1) \circ k(Z_2), \quad \forall Z_1, Z_2 \in \mathcal{T}^{1,0}_M. \tag{69}$$

Proof. Since D preserves g and h, it preserves k as well (i.e. $D_X(k(Y)) = k(D_{\bar{X}} Y)$, for any $Y \in \mathcal{T}^{1,0}_M$ and complex vector field X). From $k(e) = \mu e$,

$$\mu D_Z e = D_Z (k(e)) = k\left(\bar{\partial}_{\bar{Z}} e\right) = 0, \quad \forall Z \in \mathcal{T}^{1,0}_M$$

where we used that $e \in \mathcal{T}_M$. Thus, $D(e) = 0$, $R^D_{Z_1, \bar{Z}_2}(e) = 0$ and, from the second tt^*-equation,

$$[\phi_{Z_1}, \phi^\dagger_{\bar{Z}_2}](e) = 0, \quad \forall Z_1, Z_2 \in \mathcal{T}^{1,0}_M, \tag{70}$$

where $\phi_X(Y) = -X \circ Y$ is the Higgs field and $\phi^\dagger_{\bar{Z}} = k\phi_Z k$ is the h-adjoint of ϕ. Relation (70) implies (69) (easy check) and also $\phi^\dagger_{\bar{Z}} = \bar{\mu}\phi_{k(Z)}$, hence

$$[\phi_{Z_1}, \phi^\dagger_{\bar{Z}_2}] = 0, \quad \forall Z_1, Z_2 \in \mathcal{T}^{1,0}_M. \tag{71}$$

From (71) and the second tt^*-equation again, we obtain that D is flat. Consider a local frame $\{Z_1, \cdots, Z_n\}$ of $T^{1,0}M$, formed by D-parallel (holomorphic) vector fields. Remark that $k(Z_i)$ are also D-parallel and, in particular, holomorphic. We claim that $[Z_i, Z_j] = 0$. This follows from the first tt^*-equation, which implies

$$0 = (\partial^D \phi)_{Z_i, Z_j}(e) = D_{Z_i}(\phi_{Z_j})(e) - D_{Z_j}(\phi_{Z_i})(e) + [Z_i, Z_j] = [Z_i, Z_j],$$

where we used $D(Z_i) = D(e) = 0$. Since $g(Z_i, Z_j)$ is constant (because $D(g) = 0$, $D(Z_i) = 0$) and Z_i are holomorphic and commute, $\{Z_1, \cdots, Z_n\}$ is the basis of fundamental vector fields associated to a flat coordinate system $(t^1, \cdots, t^n)$ for g. Also, $D^{(1,0)}$ coincides with the Levi-Civita connection ∇ of g. Thus Z_i and $k(Z_i)$ are ∇-parallel, hence k is constant in the coordinate system $(t^1, \cdots, t^n)$. It remains to show that $\circ$ is also constant in this coordinate system, i.e. $Z_i \circ Z_j$ is ∇-parallel, for any i, j. For this, we apply $\bar{\partial}_{\bar{Z}}$ to relation (69). We obtain, for any $Z \in \mathcal{T}^{1,0}_M$,

$$\bar{\partial}_{\bar{Z}}\left(k(Z_i \circ Z_j)\right) = \bar{\mu}\bar{\partial}_{\bar{Z}}\left(k(Z_i)\right) \circ Z_2 + \bar{\mu} Z_1 \circ \bar{\partial}_{\bar{Z}}\left(k(Z_i)\right) + \bar{\mu}\bar{\partial}_{\bar{Z}}(\circ)\left(k(Z_1), k(Z_2)\right)$$

and the right hand side of this expression is zero ($\bar{\partial}_{\bar{Z}}(\circ) = 0$ because $\circ$ is holomorphic). We proved that $Z_i \circ Z_j$ is D-parallel, hence also ∇-parallel, and the Frobenius manifold is trivial. $\square$

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